COMBINATORICS FOR
THE ELEMENTARY GRADE CLASSROM

CONTENTS

1. FUNDAMENTAL PRINCIPLES

One of the first questions that may arise from the given topic is "What is combinatorics?" "Combinatorics" is the name for the study of the methods of counting.

There are a few basic principles upon which much of combinatorics is based. They are: the Multiplication Principle of Counting, the Addition Principle of Counting, and the Subtraction Principle of Counting.

The Multiplication Principle of Counting states: Suppose a procedure can be divided into a first step and a second step, and that each distinct way of performing step one followed by step two results in a different outcome of the procedure. Suppose also that (1) there are m ways of performing step one, and then (2) there are n ways of performing step two, then the number of possible outcomes of the procedure is the product $m \times n$.

The Addition Principle of Counting is used when you have a choice of two methods of performing a procedure; then the number of ways of performing the procedure is found by adding the number of ways using the first method and the number of ways using the second method. The principle thus states: Suppose Task 1 can be done in m ways and Task 2 can be done in n ways. If there is no way of doing both tasks at once, then the number of ways of doing Task 1 or Task 2 is the sum $m + n$.

The Subtraction Principle of Counting is used if there is double counting involved. The principle states: If there are k ways of doing both tasks at once, then the number of ways of doing Task 1 or Task 2 is $m + n - k$.

2. COMBINATIONS UNIT

Lesson 1 : In-Class Examples

Example 1. *Suppose Briana wakes up in the morning and finds that she has three pairs of shorts and four shirts, see Figure 1. How many ways can she dress for school?*

Her shorts are tan, white, and blue. Her shirts are green, yellow, orange, and red. How many possible combinations of shorts and shirts are there?

Figure 1: Shorts and Shirts

First, Briana must choose a pair of shorts. There are three ways of doing this. Then, she must choose a shirt. There are four ways of doing this. We can use a tree diagram to visualize the two step procedure, see Figure 2.

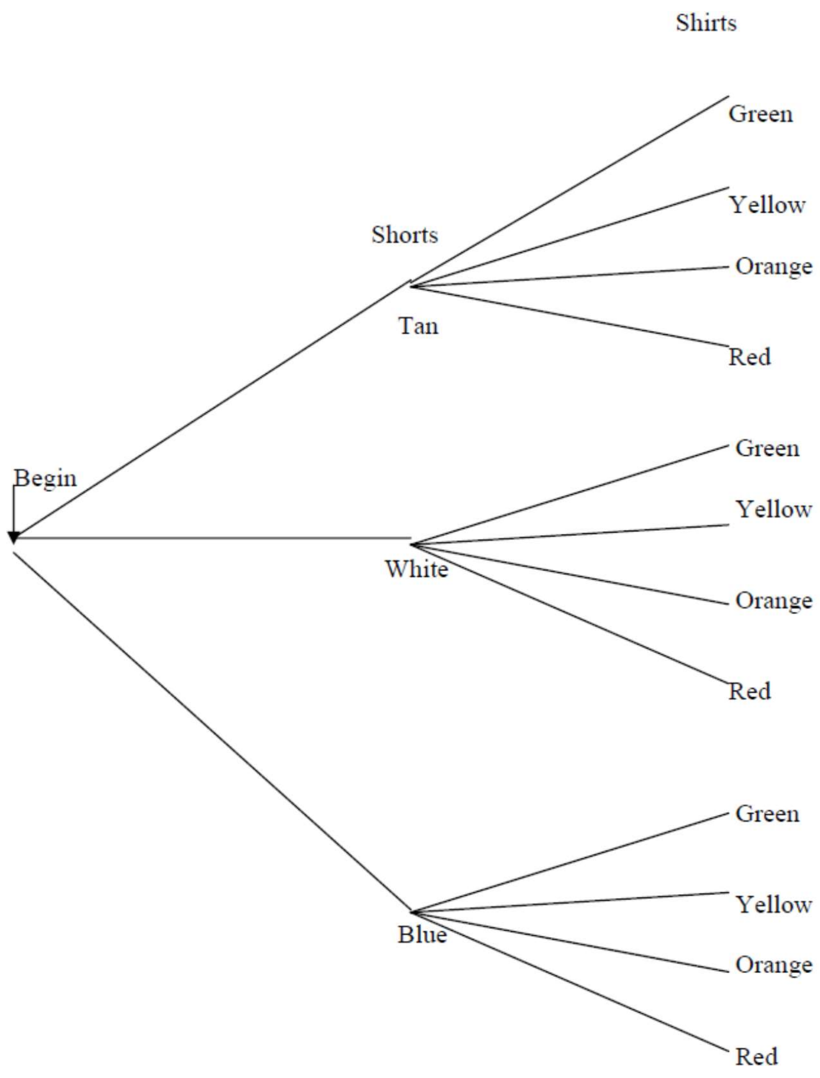

Figure 2: Tree Diagram of Shorts and Shirts

We see that there are three ways of completing STEP 1, represented by the first level branches in the tree diagram. Then for each branch in the first level, there are four second level branches, corresponding to the four ways of completing STEP 2. The points on the extreme right are leaves on the tree; these represent the ways of performing step one followed by step two. The number of leaves is the product of the number of first level options times the number of second level options: $3 \times 4 = 12$.

Example 2. *Suppose you could have a ham, turkey, or roast beef sandwich for lunch. You could also have chocolate milk or strawberry milk. How many possible combinations of a sandwich and a drink are there?*

What is Step 1? Step 1 is to list the sandwiches: HAM, TURKEY, and ROAST BEEF.

What is Step 2? For each kind of sandwich, draw branches for the kinds of drink you could have: chocolate milk and strawberry milk, see Figure 3.

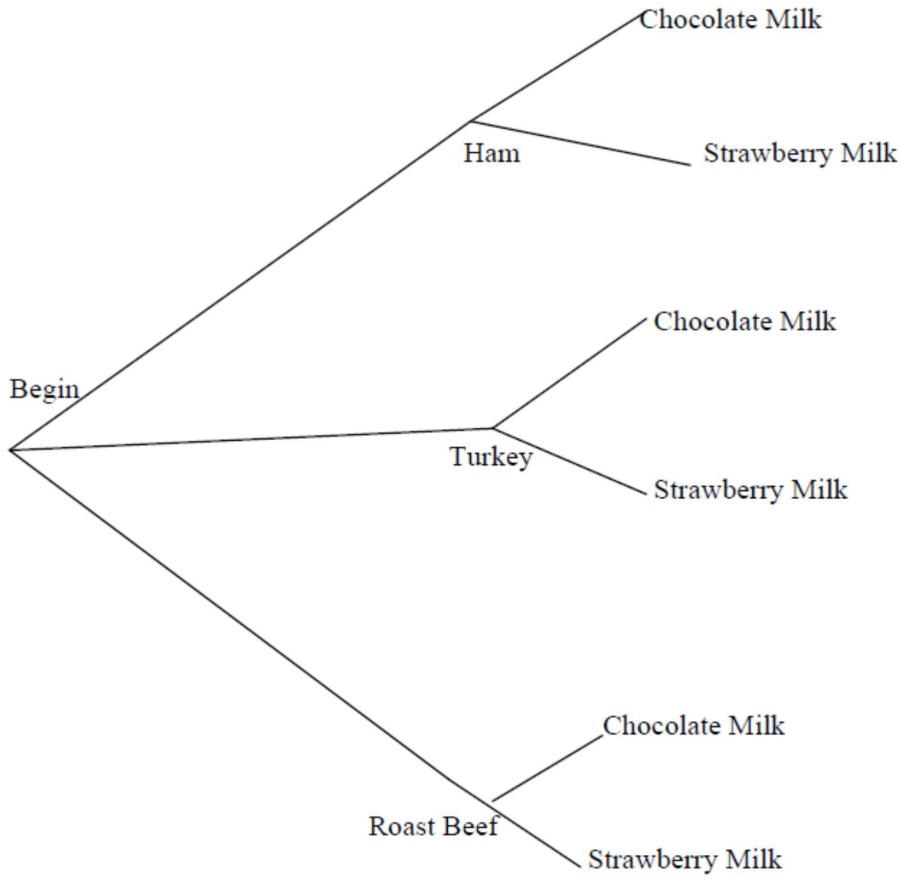

Figure 3: Tree Diagram Sandwich and Milk

So, there are six possible combinations for a sandwich and milk: $2 \times 3 = 6$.

Example 3: *Shaina is going to a local bicycle shop to purchase a bicycle. She may choose a five speed, ten speed, or fifteen speed. She may choose a purple, red, or blue bike. How many possible combinations are there?*

What is Step 1? Step 1 is to list the bicycle speed choices: FIVE, TEN, and FIFTEEN.

What is Step 2? Step 2 is to draw branches for the color choices: PURPLE, RED, and

BLUE, see Figure 4.

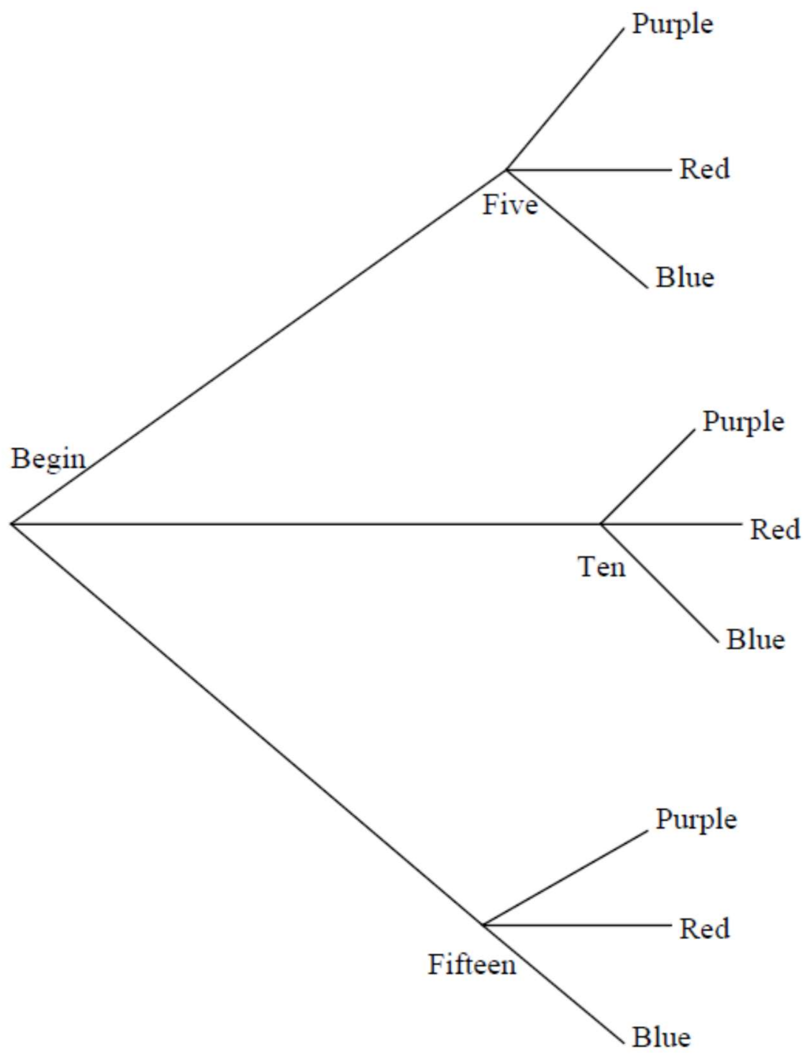

Figure 4: Tree Diagram Bicycle and Color

So, there are 9 possible combinations: $3 \times 3 = 9$.

Homework Exercises: 1. Combinations

Directions: Make a tree diagram to show all the combinations. Tell how many combinations are possible.

1. The local ice cream parlor offers three milk shake flavors: vanilla, strawberry, and chocolate. They come in three sizes: small, medium, and large. How many possible combinations are there?

2. The Kountry Kitchen restaurant offers the following side order choices: tater tots, french fries, and onion rings. They come in two sizes: small and large. How many possible combinations are there?

3. The deli at the mall offers the following food choices: hamburger, hotdog, and pizza. They offer the following drink choices: water, milk, soda, and juice. How many possible combinations are there?

4. Kourtney goes to the Italian Eatery to order a pizza. She may choose from the following pizza toppings: cheese, pepperoni, and hamburger. She must choose a small, medium, or large size. How many possible combinations are there?

5. While playing a game you must toss a coin and roll a die. When you toss a coin you can get heads or tails. When you roll a die you can get 1, 2, 3, 4, 5, or 6. How many possible combinations are there?

Lesson 2. Three or More Steps

We have worked several examples of two step combinations. However, combinations can be done in three or more steps by using the same procedure as before – tree diagrams and multiplying.

Example 1*: Flip a coin – a penny. There are two possible outcomes, heads and tails, see Figure 5.*

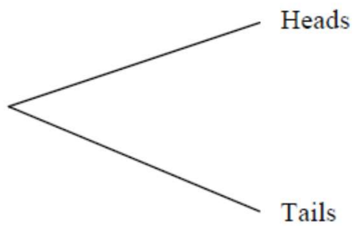

Figure 5. Penny

Now add a coin-a nickel. Flip a penny and a nickel. There are four possible outcomes, see Figure 6.

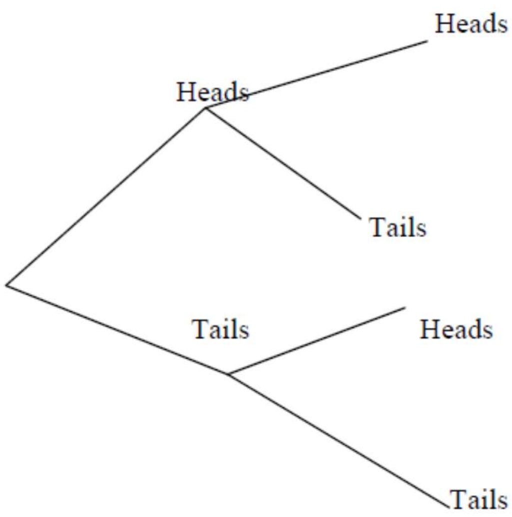

Figure 6. Nickel

Now add a dime. At this point a tree diagram becomes rather complicated. In fact, it becomes impractical. Therefore it is best to apply the multiplication principle. There are two penny outcomes, two nickel outcomes, and 2 dime outcomes: $2 \times 2 \times 2 = 8$.

Now add a quarter. There are two penny outcomes, two nickel outcomes, two dime outcomes, and 2 quarter outcomes: $2 \times 2 \times 2 \times 2 = 16$.

Example 2. *Let's refer back to Briana's clothing choices – three pair of shorts and four shirts. Let's add a third choice – shoes. She may choose sandals, flip flops, or tennis shoes. How many possible combinations are there?*

There are three short outcomes, four shirt outcomes, and three shoe outcomes: $3 \times 4 \times 3 = 36$.
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Example 3. *Ernie has twelve coins. In how many different ways can he choose the twelve coins?*

There are two outcomes for coin number 1, two outcomes for coin number 2, two outcomes for coin number 3, two outcomes for coin number 4, two outcomes for coin number 5, two outcomes for coin number 6, two outcomes for coin number 7, two outcomes for coin number 8, two outcomes for coin number 9, two outcomes for coin number 10, two outcomes for coin number 11, and 2 outcomes for coin number 12:
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 4{,}032$.

NOTE: Third grade children will need to use a calculator to multiply numbers this large.

Example 4. *Suppose a room has five lamps. Each lamp can be ON or OFF. How many different combinations of lighting the room are there?*

There are two lamp 1 outcomes, two lamp 2 outcomes, two lamp 3 outcomes, two lamp outcomes, and two lamp 5 outcomes: $\times 2 \times 2\times 2 \times 2\times 2 = 32$.

Homework Exercises: **2. Three or More Steps**

Directions: Use the multiplication principle to solve each of the following.

1. Katie is getting dressed for school. She may choose from four pairs of pants, two shirts, and six pairs of shoes. How many different ways could she get dressed?

2. Shaina goes to McDonald's for breakfast. She may choose from three kinds of biscuits: ham, sausage, or bacon. She may choose a side order: hash browns or fruit. She may choose a drink: milk, juice, or coffee. How many combinations of biscuit, side order, and drink are there?

3. During P.E., Monday through Friday, Kayla may choose between four sports: basketball, soccer, softball, and tennis. She must also choose between warm–up exercises: running or jumping rope. How many combinations of days, sports, and warm-ups does she have?

Lesson 3. Further Applications of the Multiplication Principle

Example 1. *In the state of Virginia, how many different license plates can be made if each plate is to display three letters followed by three numbers?*

Step 1: Question: How many letters in the English alphabet?
Answer: 26

So, there are 26 choices for the first letter, 26 choices for the second letter, and 26 choices for the third letter: $26\times 26\times 26$.

Step 2: Question: How many digits are used for numbers?
Answer: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9

There are 10 digits. Those ten digits are used to make all numbers. So, there are 10 choices for the first number, 10 choices for the second number, and 10 choices for the third number: $10\times10\times10$.

Step 3: Put the two steps together: $26\times 26\times 26\times 10\times 10\times 10 = 17{,}576{,}000$ possible outcomes.

Note: Third grade students will need to use a calculator to multiply these numbers.

Example 2. *A dance class has 9 boys and 11 girls. How many ways can the teacher choose a couple consisting of 1 boy and 1 girl to demonstrate the chicken dance?*

Step 1: The teacher must choose a boy. There are 9 ways of choosing a boy.
Step 2: The teacher must choose a girl. There are 11 ways of choosing a girl.
Step 3: Apply the multiplication principle: $9\times 11 = 99$.

Next suppose that the teacher may dress the girl in a dress, jeans, or a chicken suit. The teacher may dress the boy in jeans or a chicken suit. How many ways can she choose to dress the couple?

Step 1: Choose a boy. There are 9 ways to choose a boy.
Step 2: Choose a girl. There are 11 ways to choose a girl.
Step 3: Dress the boy. There are 2 ways to dress the boy.
Step 4: Dress the girl. There are 3 ways to dress the girl.
So, the number of possibilities is: $9\times 11\times 2\times 3 = 594$.

Example 3. *How many different two digit even numbers are there?*

Step 1: Question: How many ways do we have of choosing the first digit?
Answer: There are 9 ways of choosing the first digit because there are 10 digits and you can't use a zero as the first digit.

Step 2: Question: How many ways do we have of choosing the second digit?
Answer: There are 5 ways of choosing the second digit. It must be a 0, 2, 4, 6, or 8 if the number is even.

Step 3: Apply the multiplication principle: $9\times 5 = 45$.

Homework Exercises: 3. Multiplication Principle

Directions: Use the multiplication principle to solve the following problems.

1. A family consists of a mother, father, 2 girls, and 3 boys. How many different ways can the family choose one girl to wash the dishes and one boy to dry the dishes?

2. A family consists of a mother, father, 2 girls, and 3 boys. How many different ways can a boy, a girl, and a parent go shopping?

3. Katie is taking a science test. How many different ways can she answer all the questions on the test if the test has 10 true-false questions?

4. How many possible 5-digit zip codes are there?

5. How many 4-digit numbers are there that are odd?

Lesson 4. Addition Principle of Counting

As we have seen, the multiplication principle applies to procedures consisting of anumber of steps, or tasks, each of them to be carried out in order. Our next example illustrates a second fundamental principle of counting; this principle applies to procedures where there are a number of tasks, but only one of them is to be carried out.

Example 1. *A P.E. class consists of: 9 girls from Mrs. McFaddin's class, 8 boys from Mrs. McFaddin's class, 10 girls from Mrs. Richardson's class, and 11 boys from Mrs. Richardson's class. How many ways are there of choosing a girl from Mrs. McFaddin's class or a boy from Mrs. Richardson's class?*

This problem is different than the other problems we have been solving. Now instead of performing both of two tasks we are to perform only one or the other of them.There are 9 ways of selecting a girl from Mrs. McFaddin's class, and 11 ways of selecting a boy from Mrs. Richardson's class. We need to carry out only one of these tasks. To find the number of ways of selecting one student, we put the 9 girls together with the 11 boys, to obtain$9 + 11 = 20$ possibilities for choosing one student according to these rules.

Example 2. *John will draw one card from a standard deck of playing cards. How many ways can he draw a king or a queen?*

Step 1: Question: How many kings are there in a deck of cards?
Answer: There are 4 kings in a deck of cards.
Step 2: Question: How many queens are there in a deck of cards?
Answer: There are 4 queens in a deck of cards.

Step 3: If there are 4 kings and 4 queens in a deck of cards then John can't choose a king and a queen on the same draw. So, by applying the addition principle, there are 4 + 4 = 8 ways of choosing a king or a queen.

Example 3. *In a pack of 12 colored pencils there are 2 red, 2 green, 1 yellow, 2 blue,1 black, 1 brown, 1 gray, 1 purple, and 1 orange pencil. In one draw, how many ways can a student choose either a red or a green pencil?*

Step 1: Question: How many red pencils are there?
Answer: There are 2 red pencils.
Step 2: Question: How many green pencils are there?
Answer: There are 2 green pencils.
Step 3: Apply the addition principle of counting. The student can't choose a red and a green pencil in one choice. So, there are $2 \times 2 = 4$ ways of choosing a red or a green pencil.

Homework Exercises: 4. Addition Principle

Directions: Use the addition principle of counting to answer the following problems.

1. Kayla goes to the local pet store to choose a new pet. They have the following pets to choose from: 9 dogs, 12 cats, 24 hamsters, and 6 iguanas. How many ways are there of choosing a cat or a hamster?

2. Alex wins a prize for playing a game of Hot Seat in class. He may choose from the following prizes: 4 yo-yos, 6 slap bracelets, or 10 balls. How many ways are there of choosing a slap bracelet or a ball?

3. Chelsey goes to the store to purchase a candy bar. She may choose from 15 Hershey bars, 22 Reese cups, or 5 Snicker bars. How many ways are there of choosing a Reese cup or a Hershey bar?

4. Alyssa is making a mosaic. She may use 20 red squares, 20 green squares, 20 blue squares, 20 white squares, or 20 yellow squares. How many ways are there of choosing a red square or a blue square?

5. There are 5 flights from Bristol to Orlando with a stop in Atlanta, one direct flight from Bristol to Orlando, and 7 flights from Bristol to Orlando with a stop in Charlotte. In how many ways can the McFaddin family travel from Bristol to Orlando for their vacation?

Lesson 5. Subtraction Principle of Counting

Now, that we have had some practice with the addition principle of counting, we can carry it one step further and apply the subtraction principle.

We have already learned that when we have a choice of two methods of performing a procedure, then the number of ways of performing the procedure is found by adding the number of ways using the first method and the number of ways of using the second method. However, sometimes when you do this, double counting occurs. You must subtract in this situation.

Example 1. *Christian will draw one card from a standard deck of playing cards. How many ways can he choose a queen or a red card?*

Question: How many ways are there of choosing a queen?
Answer: There are four ways.

Question: How any ways are there of choosing a red card?
Answer: There are twenty-six ways. (There are fifty-two cards in a deck. Half of them are red and half of them are black.) Demonstrate this to the students using a deck of cards.

Question: Is it possible to draw a card that is a queen and a red card on the same draw? In other words, is it possible to draw a red queen?
Answer: Yes. There are two red queens in a deck of cards.

So, there are four ways of choosing a queen plus twenty-six ways of choosing a red card: $4 + 26 = 30$.

The problem is that we have counted the red queens twice. We counted them once in the queen count and again in the red count. Therefore, we must subtract two, because we counted the two red queens twice. So, there are thirty minus two ways of choosing a queen or a red card:$30 = 2 + 28$ ways.

Example 2. *A family consists of a mother, a father, two girl children, and three boy children. How many ways can the family choose a male or a child to take out the trash?*

Question: How many ways are there of choosing a male?
Answer: There's one father and three boy children. So, there are $1+ 3 = 4$ways of choosing a male.

Question: How many ways are there of choosing a child to take out the trash?
Answer: There are two girl children and three boy children. So, there are $2 + 3 = 5$ ways of choosing a child to take out the trash. So, there are $4 + 5 = 9$ ways of choosing a male or a child to take out the trash. The problem is that we counted the boy children twice.

Therefore we must subtract the three boy children: $9 - 3 = 6$. So, there are six ways of choosing a male or a child to take out the trash.

Example 3. *A family consists of a mother, a father, four girl children, and six boy children. How many ways can the family choose a female or a child to wash the car?*

Question: How many ways are there of choosing a female?
Answer: There's one mother and four girl children. So, there are $1 + 4 = 5$ ways of choosing a female.

Question: How many ways are there of choosing a child?
Answer: There are four girl children and six boy children. So, there are $4 + 6 = 10$ ways of choosing a child.

So, there are $5 + 10 = 15$ ways of choosing a female or a child to wash the car.

The problem is that we counted the four girl children twice. Therefore, we must subtract the four girl children: $15 - 4 = 11$. So, there are 11 ways of choosing a female or a child to wash the car.

Homework Exercises: 5. Subtraction Principle

Directions: Use the subtraction principle to solve each of the following problems.

1. You are to draw one card from a deck of 52 cards. How many ways can you choose a king or a black card?

2. You are to draw one card from a deck of 52 cards. How many ways can you choose a two, a five, or a black card?

3. In a high school math class, there are six girls in the tenth grade, eight boys in the tenth grade, five girls in the ninth grade, and four boys in the ninth grade. In how many ways can the teacher choose a girl or a tenth grader to answer a question?

4. In the above example, in how many ways can the teacher choose a ninth grader or a boy to call the roll?

3. PERMUTATIONS UNIT

In this unit we will examine a concept in counting called permutations. The number of ways that you can change the order of a set of things is called the number of permutations of that set of things.

Lesson 6. In Class Exercises

Example 1. *How many ways can you arrange the letters in the word STOP?*

Words that start with "S": STOP STPO SOTP SOPT SPTO SPOT

Words that start with "T": TSOP TSPO TOSP TOPS TPSO TPOS

Words that start with "O": OSTP OSPT OTSP OTPS OPST OPTS

Words that start with "P": PSTO PSOT PTSO PTOS POST POTS

There are 24 ways to order the letters in STOP. You might ask if there is a rule to follow. The answer is yes. The rule is as follows:

1. There are 4 ways to pick the first letter. It can be "S", "T", "O", or "P".
2. After you pick the first letter there are 3 ways to pick the second letter.
3. After you pick the first two letters, there are 2 ways to pick the third letter.
4. After picking the first three letters, there is only one letter left to pick.

So, the number of ways to order the letters in "STOP" is $4\times3\times 2\times1 = 24$ ways.

Example 2. *How many ways can you arrange the letters in the word* BOY?

Words that start with "B": BOY BYO

Words that start with "O": OBY OYB

Words that start with "Y": YOB YBO

There are 6 ways to order the letters in BOY. If you follow the rule, you will get
$3\times 2\times1 = 6$.

Example 3. *How many ways can you arrange the letters in* HONAKER?

1. There are 7 ways to pick the first letter.
2. After you pick the first letter, there are 6 ways to pick the second letter.
3. After you pick the second letter, there are 5 ways to pick the third letter.

4. After you pick the third letter, there are 4 ways to pick the fourth letter.
5. After you pick the fourth letter, there are 3 ways to pick the fifth letter.
6. After you pick the fifth letter, there are 2 ways left to pick the sixth letter.
7. After you pick the sixth letter, there's one way to pick the seventh letter.

So, there are $7 \times 6\times 5\times 4\times 3\times 2\times 1 = 5{,}040$ ways to arrange the letters in HONAKER .

NOTE: Third grade students will need to use a calculator to multiply numbers that are this large.

Example 4. *How many ways can you arrange the letters in* BASKET?

Now that you understand the process that is involved, we can move straight to the rule.
So, there are $6\times 5\times 4\times 3\times 2\times 1 = 720$ ways to arrange the letters in the word BASKET.

Homework Exercises: 6. Permutations

1. How many different ways can you arrange the letters in the word SUN?

2. How many different ways can you arrange the letters in the word GIRL?

3. How many different ways can you arrange the letters in the word TIGER?

4. How many different ways can you arrange the letters in the word STUDENT?

5. How many different ways can you arrange the letters in your name?

Lesson 7. More Permutations

Now, let's suppose that you only want to choose a few letters out of a word. For example, you only want to choose 2 letters out of the word "TABLE". Here are all the ways to pick them:

TA TB TL TE AT AB AL AE BT BA

BL BE LT LA LB LE ET EA EB EL

There are 20 pairs. You may want to know if there is an easier method than writing them all down. Here it is:

1. There are 5 ways to choose the first letter.

2. After you choose the first letter, there are 4 ways to choose the second letter.
So, the number of two letter permutations of the five letter word "TABLE" is: $5\times 4 = 20$.

General Rule:

If you have a word with "n" letters in it, then:

- to select 2 letters, the number of permutations is $n \times (n-1)$
  Example: TABLE, $5\times (5-1) = 5\times 4 = 20$
- to select 3 letters, the number of permutations is $n \times (n-1) \times (n-2)$
  Example: TABLE, $5\times (5-1)(5-2) = 5\times 4\times 3 = 60$
- to select 4 letters, the number of permutations is
  $n \times (n-1) \times (n-2) \times (n-3)$
  Example: TABLE, $5\times (5-1)(5-2)(5-3) = 5\times4\times3\times 2 = 120$

Example 1: *Suppose we want to find the number of ways to arrange the three letters of the word DOG in different two-letter groups where DO is different from OD and there are no repeated letters.*
Because order matters, we're finding the number of permutations of size two that can be taken from a set of size three. We could list them: DO, DG, OD, GD, OG, and GO. Or we could apply the rule:$3\times (3-1) = 3\times 2 = 6$.

Example 2: *Now, let's suppose that we have ten letters and want to make groupings of four letters.*

It's harder to list all those permutations. So, it is best to apply the rule:
$10\times (10-1) \times (10-2) \times (10-3) = 10\times 9\times8\times 7 = 5{,}040$.

Example 3: *If the five letters, a, b, c, d, and e are put into a hat, in how many different ways could you draw out three letters?*

Here you must think, I have a set of five and I want to choose three. So, you apply the rule: $5\times (5-1) \times (5-2) = 5\times 4\times 3 = 60$. There are 60 different ways to choose a set of three out of the five letters.

Example 4: *You have 21 students in your room. You want to put your students in groups of three. You want the students in order from left to right. In how many different ways could you do this?*

Here you must think, I have 21 students, and I want to choose groups of three. So, you apply the rule: $21\times(21-1)\times(21-2)=21\times 20\times 19=7{,}980$ways.

Homework Exercises: 7. More Permutations

Directions: Use the permutations rule to solve the following problems.

1. You go to the local ice cream shop. They have 20 different flavors of ice cream. You want to make a cone that has two scoops of ice cream on it, a top layer and a bottom layer. In how many different ways could you choose two scoops from the 20 flavors?

2. Suppose you want to find the number of ways to arrange the five letters in the word PIZZA in different three-letter groups. In how many different ways can we do this?

3. You go back to the local ice cream shop. Today they only have fourteen flavors. You want to make a cone that has three scoops of ice cream on it, a top layer, a middle layer, and a bottom layer. In how many different ways could you choose three scoops from the 14 flavors?

4. Suppose we want to find the number of ways to arrange the six letters in the word TIGERS in different three- letter groups. In how many different ways can we do this?

Lesson 8. Factorial Representation of Permutations

Today we are looking at the symbol "!". The symbol, "!", in permutations means to multiply. For example, 5! means $5\times 4\times 3\times 2\times 1=120$. 3! means$3\times 2\times 1=6$, and

$6!=6\times 5\times 4\times 3\times 2\times 1=720$.

Example 1: *Five different books are on a shelf. In how many different ways can you arrange them?*

Answer: *There are 5! ways to arrange them: $5\times 4\times 3\times 2\times 1=120$.*

Example 2: *There are 6 letters in the word TIGERS. In how many different ways can you arrange them?*

Answer: There are 6! ways to arrange them: $6\times 5\times 4\times 3\times 2\times 1=720$.

Example 3: *There are 6! permutations of the 6 letters of the word SQUARE. In how many of them is R the second letter?*

Answer: __ R __ __ __ __

Let R be the second letter. Then there are 5 ways to fill the first spot, 4 ways to fill the third spot, 3 ways to fill the fourth spot, 2 ways to fill the fifth spot, and 1 way to fill the last spot. There are 5! such permutations: 5× 4× 3× 2×1 = 120ways.

Example 4: *There are 5! permutations of the word VIDEO. In how many of them is D the second letter?*

Answer: __ D__ __ __

Let D be the second letter. Then there are 4 ways to fill the first spot, 3 ways to fill the third spot, 2 ways to fill the fourth spot, and 1 way to fill the last spot. There are 4! Such permutations: 4×3× 2×1 = 24ways.

Example 5: *There are 3! permutations of the word CAT. In how many of them is C the last letter?*

Answer: __ __ C

Let C be the last letter. Then there are 2 ways to fill the first spot and 1 way to fill the second spot. There are 2! such permutations: 2×1 = 2ways.

Example 6: *There are 7! permutations of the word STUDENT. In how many of them is E the fifth letter?*

Answer: __ __ __ __ E __ __

Let E be the fifth letter. Then there are 6ways to fill the first spot, 5 ways to fill the second spot, 4 ways to fill the third spot, 3 ways to fill the fourth spot, 2 ways to fill the sixth spot, and 1 way to fill the last spot. There are 6! such permutations:

6× 5× 4× 3× 2×1 = 720 ways.

Homework Exercises: 8. Factorials

Directions: Use factorials to solve the following problems.

1. There are 4! permutations of the 4 letters of the word BLUE. In how many of them is the letter U the second letter?

2. There are 6! permutations of the 6 letters of the word ORANGE. In how many of them is the letter O the last letter?

3. There are 8! permutations of the 8 letters of the word FLAVORED. In how many of them is the letter V the third letter?

4. There are 10! permutations of the 10 letters of the word MICROWAVES. In how many of them is the letter W the fourth letter?

4. MIXING THINGS UP

In this lesson students will discover, through interactive play on the computer, that a combination is a set of objects in which order is not important.

Puzzles

Remind students that a combination is a set of objects in which order is not important. Review the puzzles; Football Strips, Ice Cream Cones, and Cars. Allow students to share the puzzles that they designed. Discuss the combinations of objects as they are presented.

Lesson 9. Permutations and Combinations

In this lesson we will be combining the idea of permutations and combinations. Remember from our prior lessons that the order of the objects is important in permutations. In other words *abc* is different from *bca*. In combinations, however, we are only concerned that *a,b,*and *c* have been chosen. In other words *abc* and *bca* are the same combination. Here are all the combinations of *abcd* taken three at a time: *abc, abd, acd,* and *bcd.* There are four such combinations. We call this:

The number of combinations of four things taken three at a time.

Now, how are the number of combinations related to the number of permutations?

Let's look at the number of permutations for each combination:

abc	*abd*	*acd*	*bcd*
acb	*adb*	*adc*	*bdc*
bac	*bad*	*cad*	*cbd*
bca	*bda*	*cda*	*cdb*
cab	*dab*	*dac*	*dbc*
cba	*dba*	*dca*	*dcb*

Each column is the 3! permutations of that combination. But they are all one combination because order does not matter. Hence, there are 3! times as many permutations as combinations. Therefore, in order to find the number of combinations of 4 options, choosing 3, take the number of permutations and divide by the number of permutations that each combination generates. Number of permutations of 4 options, choosing 3: $4\times3\times 2 = 24$

Number of permutations that each combination generates: $1\times 2\times 3 = 6$.

Now divide: $24 \div 6 = 4$.

Example 1: *How many combinations are there of 5 cookies taken 4 at a time*?

Number of permutations of 5 cookies, choose 4: $5\times 4\times 3\times 2 = 120$.

Number of permutations that each combination generates: $1\times 2\times 3\times 4 = 24$.

Now divide: $120 \div 24 = 5$.

Example 2: *How many combinations are there of 8 students chosen 6 at a time?*

Number of permutations of 8 students, choose 6: $8\times 7 \times 6\times 5\times 4\times3 = 20{,}160$.

Number of permutations that each combination generates: $1\times 2\times 3\times 4\times 5\times 6 = 720$.

Now divide: $20{,}160 \div 720 = 28$.

Example 3*:* *How many combinations are there of 8 dogs chosen 2 at a time?*

Number of permutations of 8 dogs, choose 2: $8\times 7 = 56$.

Number of permutations that each combination generates: $1\times 2 = 2$.

Now divide: $56 \div 2 = 28$.

Example 4: *There are 12 boys in Mrs. McFaddin's class. In how many ways can Mrs. McFaddin choose 2 boys from the 12?*

Number of permutations of 12 boys, choose 2: $12\times11 = 132$.

Number of permutations that each combination generates: $1\times 2 = 2$.

Now divide: $132 \div 2 = 66$.

Example 5: *There are 14 girls in Mrs. McFaddin's class. In how many ways can Mrs. McFaddin choose 3 girls from 14?*

Number of permutations of 14 girls, choose 3: $14\times13\times12 = 2{,}184$.

Number of permutations that each combination generates: $1\times 2\times 3 = 6$.

Now divide: $2{,}184 \div 6 = 364$.

Homework Exercises: 9. Permutations and Combinations

Directions: Use the number of permutations and the number of each permutation that each combination generates to solve the following problems.

1. There are 6 toys from which to choose and you are to choose 2 of them. In how many ways can you choose 2 toys from the 6 toys?

2. There are 12 pizza toppings from which to choose and you are to choose 3 of them. In how many ways can you choose 3 toppings from the 12?

3. There are 4 kittens from which to choose and your mom says that you may choose 2 to take home. In how many different ways can you choose 2 kittens from the 4?

4. There are 8 toppings for your hamburger. In how many different ways can you choose 4 toppings for your hamburger?

5. There are 48 games to choose from at the store. Your dad says that you may have 3 games to take home. In how many different ways can you choose the 3 games?

Lesson 10. More Combinations

Example 1: *Let's suppose that we have 10 posters to choose from. We want to choose 3 of them to hang on the wall in our classroom. In how many different ways could we do this?*

We can think of this in terms of 10 posters, choose 3. The order does not matter.
Step 1: $10 \times 9 \times 8 = 720$
Step 2: Each of these can be arranged in 3! ways: $1 \times 2 \times 3 = 6$
Step 3: $720 \div 6 = 120$ different ways to choose the posters.

Example 2: *Let's suppose that we have 25 books to choose from. We want to choose 2 of them to check out. In how many different ways can you do this?*

Think in terms of 25 books, choose 2.
Step 1: $25 \times 24 = 600$
Step 2: $1 \times 2 = 2$
Step 3: $600 \div 2 = 300$ different ways to choose the books.

Example 3: *You are going to summer camp. You have 5 swimming suits. Your mom says that you may only take 3 of them with you. In how many different ways can you do this?*

Think in terms of 5 suits, choose 3.
Step 1: $5 \times 4 \times 3 = 60$

Step 2: 1× 2× 3 = 6

Step 3: 60 ÷ 6 = 10 different ways to choose the swimming suits.

Example 4: *You are at a salad bar. There are 35 items on the bar. You want to put 5 items on your salad. In how many different ways can you select the 5 items?*

Think in terms of 35 items, choose 5.

Step 1: 35× 34× 33× 32× 31 = 38,955,840

Step 2: 1× 2× 3× 4×5 = 120

Step 3: 38,995,840 ÷120 = 324,632 different ways.

Example 5: *You are going to the beach. You go on the internet and find that there are 14 different routes to take. You want to choose 2 different routes. In how many different ways could you do this?*

Think in terms of 14 routes, choose 2.

Step 1: 14×13 = 182

Step 2: 1× 2 = 2

Step 3: 182 ÷ 2 = 91 different ways.

Example 6: *You go to the local shoe store. They have 32 pairs of shoes in your size. In how many different ways could you choose 4 pair to take home?*

Think in terms of 32 choose 4.

Step 1: 32×31× 30× 29 = 863,040

Step 2: 1× 2× 3× 4 = 24

Step 3: 863,040 ÷ 24 = 35,9608 different ways.

Homework Exercises: 10. More Combinations

Directions: Use combinations and permutations to solve each of the following problems.

1. You want to choose a soda from the machine. There are ten different types of sodas in the machine. You want to choose three sodas. In how many different ways can you do this?

2. At the local store there are 15 different kinds of salad dressings. Your mom told you to choose 3 different kinds. In how many ways can you do this?

3. You are designing a flower arrangement. You have 22 different kinds of flowers to choose from. You need to choose 4 kinds to include in your arrangement. In how many different ways can you do this?

4. There are 7 different balls to choose from. You need to choose 5 balls. In how many different ways can you do this?

5. You have 25 channels to choose from. You may choose 5 from the 25 channels to have on your T.V. In how many different ways can you do this?

Lesson 11. Combination and Permutation Problems

Example 1: *There are 6 girls and 6 boys on Kourtney's t-ball team. In how many different ways can the coach select a team of 9 players? The team consists of 4 girls and 5 boys.*

The first step in solving this problem is to identify how many ways there are of selecting a boy. There are 6 boys and you need to choose 5. So, $6\times 5\times 4\times 3\times 2 = 720$ divided by $1\times 2\times 3\times 4\times 5 = 120$ is equal to 6.

The second step in solving this problem is to identify how many ways there are of selecting a girl. There are 6 girls and you need to choose 4. So, $6\times 5\times 4\times 3 = 360$ divided by $1\times 2\times 3\times 4 = 24$ is equal to 15.

The third step is then to multiply the number of ways of choosing a boy by the number of ways of choosing a girl: $5\times 15 = 90$ways.

Example 2: *There are 10 boys and 12 girls in Mrs. McFaddin's third grade class. She wants to select a group of 3 students from the class to work on a math project. She wants the group to consist of 2 girls and 1 boy. In how many ways can she do this?*

Step 1: How many ways can she choose the 2 girls?

$12\times 11 = 132$ divided by $1\times 2 = 2$which is equal to 66 ways

Step 2: How many ways can she choose the boy?

$10\times 1 = 10$ ways.

Step 3: Multiply the number of ways of choosing a girl by the number of ways of choosing a boy: $66\times 10 = 660$ways.

Example 3: *At a local ice cream shop there are 3 different types of cones and 52 different flavors of ice cream. In how many ways can you choose 1 cone with 2 scoops of ice cream?*

Step 1: How many ways can you choose a cone?

$3 \times 1 = 3$ ways

Step 2: How many ways can you choose 2 scoops of ice cream?

$52\times 51 = 2{,}652$ divided by $1\times 2 = 2$ equals 1,326

Step 3: Multiply the number of ways of choosing a cone by the number of ways of choosing 2 scoops of ice cream: $3\times1{,}326 = 3{,}978$ways.

Example 4*: You are packing your suitcase to go on vacation. You have 15 different shirts and 9 different pairs of shorts. You need to choose 5 shirts and 5 pairs of shorts. In how many ways can you choose the 5 outfits?*

Step 1: How many ways can you choose the shirts?

15 shirts, choose 5

$15\times14\times13\times12\times11 = 360{,}360$ divided by $1\times 2\times 3\times 4\times5 = 120$ equals 3,003

Step 2: How many ways can you choose the shorts?

9 shorts, choose 5

$9\times8\times 7 \times 6\times5 = 15{,}120$ divided by $1\times 2\times 3\times 4\times5 = 120$ equals 126

Step 3: Multiply the number of ways of choosing a shirt by the number of ways of choosing shorts: $3{,}003\div126 = 378{,}378$ways.

Homework Exercises: 11. Combination and Permutation Problems

Directions: Use your knowledge of combinations and permutations to solve each of the following problems.

1. At the local pizza place there are 8 choice for meat toppings and 10 choices for vegetable toppings. You want to choose 3 meats and 5 vegetables for your pizza. How many ways can you choose the 8 toppings?

2. Mr. Matthews has 24 boys and 26 girls in his P.E. class. He wants to choose 6 students for a volleyball team. The team will consist of 3 boys and 3 girls. In how many ways can he choose his team?

3. Mr. Hubbard has 12 Senior boys and 14 Junior boys going out for the football team. He wants to choose 8 Seniors and 3 Juniors for his team. In how many different ways can he do this?

4. I visit the local candy shop. There are 18 different types of candy bars and 10 different types of lollipops. I want to choose a bag of candy consisting of 5 bars ×of candy and 5 lollipops. In how many different ways can I do this?

5. PASCAL'S TRIANGLE

Pascal's Triangle is an arithmetical triangle that is used for some neat and clever things in mathematics. The triangle was named after Blaise Pascal who lived from 1623 – 1662.

Lesson 12a. How to Construct the Triangle

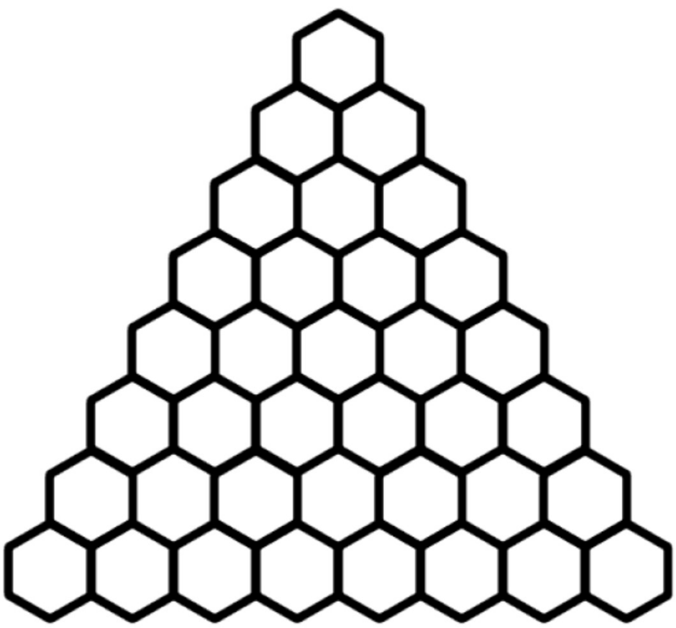
You start out with the top two rows: 1, and 1, 1, see in Figure. Then to construct each entry in the next row, you look at the two entries above it (the entry to the left and right of it). At the beginning and the end of each row, when there's only one number above it, put a 1. If you were to add the numbers above it to the left and to the right; you are adding 1 and 0, so you get 1. You can continue doing this, adding as many lines as you would like. Now, using Worksheet #1, we will practice constructing Pascal's Triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's Triangle

Worksheet #1

Discovering Patterns
Name ____________________



12b. Patterns on Pascal's Triangle

Now that we have had practice constructing the triangle, we are going to look for patterns in the triangle. Study the numbers in the triangle.

Question: What patterns do you see in the arrangement of the numbers?
Possible Answers: Natural Numbers (1, 2, 3, 4)
Triangular Numbers (1, 3, 6, 10)
Row of Ones

Question: Can you predict the next row of numbers?
Possible Answer: Yes. Continue the pattern.

Question: Add the numbers in each row. Is there a pattern in the sums of these numbers?
Answer: Yes, the sum of the rows double.

Question: Do any numbers repeat?
Answer: Yes. The ones start and end each row.

Question: Can you find a pattern in the diagonal numbers?
Possible Answers: Yes. The first diagonal row is ones. The second diagonal row is counting by ones. The third diagonal row is add 2, add 3, add 4, add 5, etc.

Now that you are aware of some of the patterns in the triangle you are going to do an activity in which you will color the odd and even numbers. Then you will look for more patterns.

Directions for Activity: Distribute a copy of Worksheet #2 to each student. Color the odd numbers green and the even numbers red. Look for a pattern.

Lesson 13a. The Mathematics in the Patterns

1. Students practice the concept of even and odd numbers by coloring odd-numbered spaces green. They should see that the triangle is outlined in green spaces because the number 1 is odd.

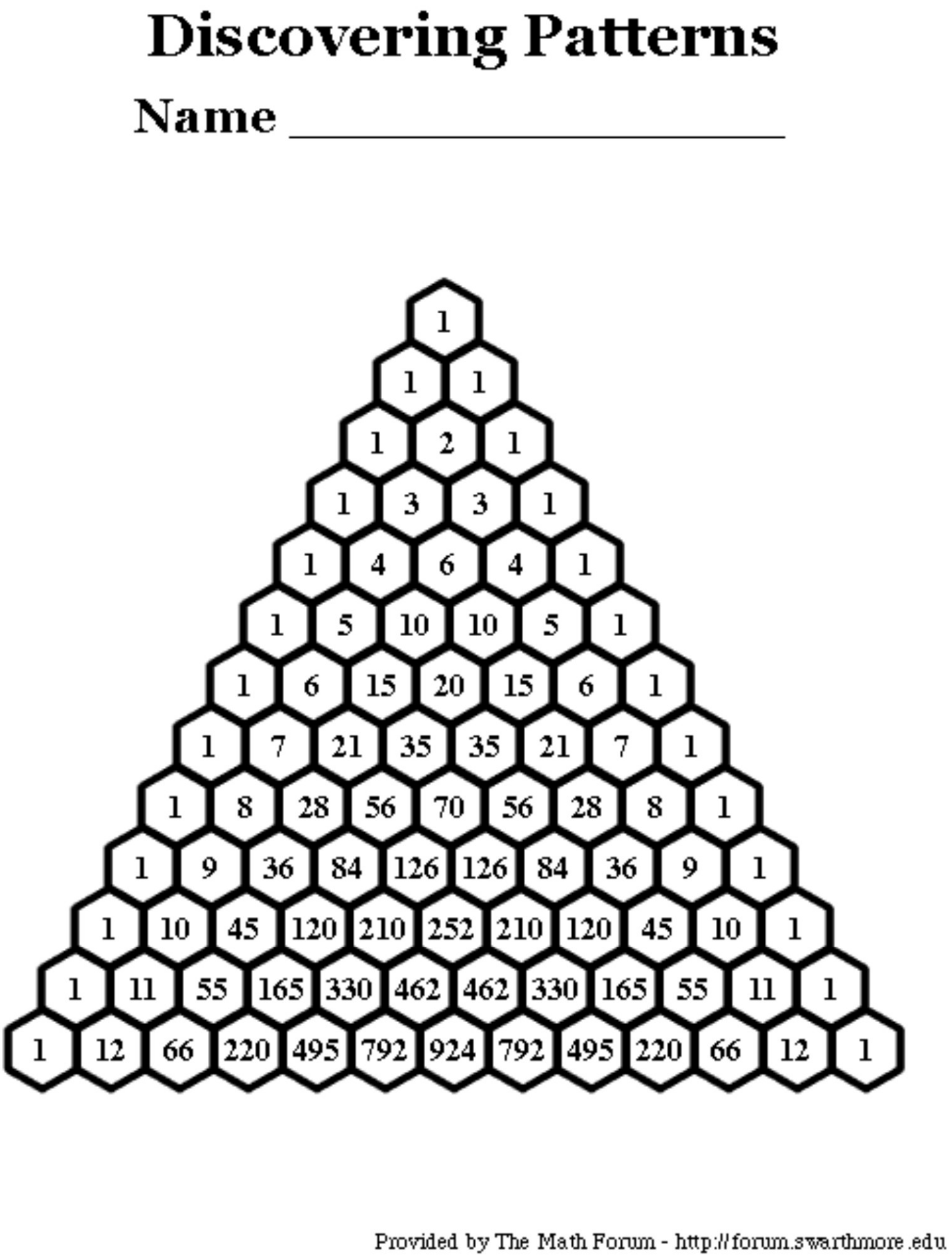
2. The Commutative Property of Addition. This property, together with the structure of Pascal's Triangle, explains the symmetry that can be observed in the colors of each row of numbers. It is not necessary for students to know or understand this property to be able to appreciate the symmetry in the coloring: if the triangle is folded through its center, spaces of the same color will fall on top of each other.

3. The sum of two odd numbers is an even number. When two green numbers are close to each other, the number below it will be red, see Figure.

4. The sum of two even numbers is an even number. When two red numbers are next to each other, the number below it will be red, see Figure 16.

5. The sum of an odd number and en even number is an odd number. When a green number is next to a red number, the number below it will be green.

Worksheet #2



Lesson 13b. Reading Pascal's Triangle

When reading Pascal's Triangle, people usually give a row number and a place in that row, beginning with row zero and place zero [10]. For instance, the number 20 appears in row 6, place 3.

Question: Where would you find the number 28?
Answer: Row 8, place 2.

Question: Where would you find the number 35?
Answer: Row 7, place 3 and 4.

Question: Where would you find the number 56?
Answer: Row 8, place 3 and 5.

Question: Where would you find the number 2?
Answer: Row 2, place 1.

Question: Where would you find the number 210?
Answer: Row 10, place 4 and 6.

Homework Exercises: 13. Pascal's Triangle

Directions: Use Worksheet #2, see Figure 15, to answer the following questions.

1. Where would you find the number 6?

2. Where would you find the number 252?

3. Where would you find the number 70?

4. Where would you find the number 495?

5. Where would you find the number 55?

Lesson 14. Pascal's Triangle and Combinatorics

In this lesson we are going to learn how to use Pascal's Triangle to find combinations.

Example 1. Let's say you have five books on a shelf, and you want to know how many ways you can pick two of them to read. It doesn't matter which book you read first. So, this problem amounts to the question "how many different ways can you pick two objects from a set of five objects?" You need to think 5 choose 2.

The answer is the number in the second place in the fifth row, see Figure.

```
              1
            1   1
          1   2   1
        1   3   3   1
      1   4   6   4   1
    1   5  10  10   5   1
  1   6  15  20  15   6   1
1   7  21  35  35  21   7   1
```

Example 2. Let's say that you have six coats in your closet, and you want to know how many ways you can pick three of them to wear. It doesn't matter which coat you wear first. So, this problem amounts to "how many different ways can you pick three objects from a set of six?" You need to think 6 choose 3.

The answer is the number in the third place in the sixth row.

Example 3. You go to the local pizza parlor. Mariano tells you that he has 9 different pizza toppings. Today's special is a large pizza with three toppings. In how many different ways can you choose the three toppings. In other words "how many ways can you pick three objects from a set of nine?" You need to think in terms of 9 choose 3.

The answer is the number in the third place of the ninth row, see Figure.

```
                    1
                  1   1
                1   2   1
              1   3   3   1
            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
    1   8  28  56  70  56  28   8   1
  1   9  36  84 126 126  84  36   9   1
```

Example 4. In the problem above, in how many different ways can you choose a pizza with four toppings? In other words "how many ways can you pick four objects from a set of nine?" You need to think in terms of 9 choose 4.

The answer is the number in the fourth place of the ninth row.

Homework Exercises: 14. Pascal and Combinatorics

Directions: Use Worksheet #2 to answer the following questions.

1. Mrs. McFaddin has 12 girls in her room. In how many different ways can she choose a group of 3 girls from her class?

2. Mrs. McFaddin has 10 boys in her class. In how many different ways can she choose a group of 3 boys from her class?

3. The concession stand has 7 different types of snacks. In how many different ways can you choose 2 snacks from the concession stand?

4. The local ice cream shop has 11 flavors of ice cream. In how many different ways can I choose 3 scoops of ice cream?

5. The local soda shop has 9 different flavors of soda. In how many different ways can I choose 4 flavors of soda?

Lesson 15. Using Pascal's Triangle to Find the Sum of Combinations

Let's go back to Mariano's Pizza Place to look at finding the sum of combinations. Let's start by asking "How many different 1-topping pizzas can you order when choosing from 10 toppings?" Think 10 choose 1. By using Pascal's triangle, we find place 1 in row 10. Thus, the answer is 10; see Figure

```
                    1
                  1   1
                1   2   1
              1   3   3   1
            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
    1   8  28  56  70  56  28   8   1
  1   9  36  84  126 126 84  36   9   1
1  10  45  120 210 252 210 120 45  10   1
```

Now, let's ask the question, "How many different two- topping pizzas can you order when choosing from ten toppings?" Think 10 choose 2. By using Pascal's Triangle, we find place 2 in row 10. Thus, the answer is 45, see Figure.

Let's now ask the question, "How many different four-topping pizzas can you order when choosing from ten toppings?" Think 10 choose 4. By using Pascal's Triangle, we find place 4 in row 10. Thus the answer is 210.

You can now begin to see the pattern to what we are doing. We can now use Pascal's Triangle to answer the question of, "What's the total number of different pizza combinations that can be made given a choice of 10 toppings?" We look at row 10 and find the following:

1 pizza with no toppings

10 different pizzas with 1 topping

45 different pizzas with 2 toppings

120 different pizzas with 3 toppings

210 different pizzas with 4 toppings

252 different pizzas with 5 toppings

210 different pizzas with 6 toppings

120 different pizzas with 7 toppings

45 different pizzas with 8 toppings

10 different pizzas with 9 toppings

1 pizza with 10 toppings

To find the sum of the numbers in row 10, see Figure:

$1+10 + 45 +120 + 210 + 252 + 210 +120 + 45 +10 +1 = 1{,}024$

Example 1. *What is the total number of jewelry combinations that can be made given a choice of 7 pieces of jewelry? To answer this question let's refer to row 7 of Pascal's Triangle?*

Row 7 tells us the following:

1 jewelry combination with 0 pieces of jewelry

7 jewelry combinations with 1 piece of jewelry

21 jewelry combinations with 2 pieces of jewelry

35 jewelry combinations with 3 pieces of jewelry

35 jewelry combinations with 4 pieces of jewelry

21 jewelry combinations with 5 pieces of jewelry

7 jewelry combinations with 6 pieces of jewelry

1 jewelry combination with 7 pieces of jewelry

To find the sum of the combinations add the numbers in row 7:

1+ 7 + 21+ 35 + 21+ 7 +1 = 128

Example 2. *What is the total number of different sundae topping combinations that can be made given a choice of 8 toppings?*

To answer this question refer to row 8 of Pascal’s Triangle, see Figure.

```
                    1
                  1   1
                1   2   1
              1   3   3   1
            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
    1   8  28  56  70  56  28   8   1
  1   9  36  84  126 126 84  36   9   1
1  10  45  120 210 252 210 120 45  10   1
```

Row 8 tells us the following:

1 sundae with no toppings

8 sundaes with 1 topping

28 sundaes with 2 toppings

56 sundaes with 3 toppings

70 sundaes with 4 toppings

56 sundaes with 5 toppings

28 sundaes with 6 toppings

8 sundaes with 7 toppings

1 sundae with 8 toppings

To find the sum of all the combinations add the numbers in row 8:

1+ 8 + 28 + 56 + 70 + 56 + 28 + 8 +1 = 256

Homework Exercises: 15. Sum of Combinations

Directions: Use Pascal's Triangle and the sum of combinations to solve the following problems.

1. What is the total number of different picture combinations that can be made given a choice of 4 pictures?

2. What is the total number of different pizza combinations that can be made given a choice of 6 toppings?

3. What is the total number of different sundae combinations that can be made given a choice of 5 toppings?

4. What is the total number of different hamburger toppings that can be made given

a choice of 9 toppings?

5. What is the total number of fingernail polish combinations that can be made given a choice of 12 colors?

6. SUMMARY

Since the Combinatorics Units were designed to introduce third grade students to combinations and permutations, the author thought that it would be appropriate toteach the units to her third grade class.