Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. Suppose x_1, x_2, x_3 are the roots of polynomial

$$P(x) = x^3 - 6x^2 + 5x + 12.$$

The sum $|x_1| + |x_2| + |x_3|$ is
(A): 4 (B): 6 (C): 8 (D): 14 (E): None of the above.

Solution. The choice is (C).

Question 2. How many pairs of positive integers (x, y) are there, those satisfy the identity

$$2^x - y^2 = 1?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Solution. The choice is (A).

Question 3. Suppose $n^2+4n+25$ is a perfect square. How many such non-negative integers n's are there?
(A): 1 (B): 2 (C): 4 (D): 6 (E): None of the above.

Solution. The choice is (B).

Question 4. Put

$$S = 2^1 + 3^5 + 4^9 + 5^{13} + \cdots + 505^{2013} + 506^{2017}.$$

The last digit of S is
(A): 1 (B): 3 (C): 5 (D): 7 (E): None of the above.

Solution. The choice is (E).

Question 5. Let a, b, c be two-digit, three-digit, and four-digit numbers, respectively. Assume that the sum of all digits of number $a + b$, and the sum of all digits of $b + c$ are all equal to 2. The largest value of $a + b + c$ is
(A): 1099 (B): 2099 (C): 1199 (D): 2199 (E): None of the above.

Solution. The choice is (E).

Question 6. Find all triples of positive integers (m, p, q) such that

$$2^m p^2 + 27 = q^3, \quad \text{and } p \text{ is a prime.}$$

Solution. By the assumption it follows that q is odd. We have

$$2^m p^2 = (q - 3)(q^2 + 3q + 9).$$

Remark that $q^2 + 3q + 9$ is always odd. There are two cases:
Case 1. $q = 2^m p + 3$. We have

$$q^3 = (2^m p + 3)^3 > 2^m p^2 + 27,$$

which is impossible.

Case 2. $q = 2^m + 3$. We have

$$q^3 = 2^{3m} + 9 \times 2^{2m} + 27 \times 2^m + 27 = 2^m p^2 + 27,$$

which implies

$$p^2 = 2^{2m} + 9 \times 2^m + 27.$$

If $m \geq 3$, then $2^{2m} + 9 \times 2^m + 27 \equiv 3 \pmod 8$, but $p^2 \equiv 1 \pmod 8$. We deduce $m \leq 3$. By simple computation we find $m = 1, p = 7, q = 5$.

Question 7. Determine two last digits of number

$$Q = 2^{2017} + 2017^2.$$

Solution. We have

$$\begin{aligned} 2^{2017} &= 2^7 \times \left(2^{10}\right)^{201} = 128 \times 1024^{201} \\ &\equiv 128 \times (-1)^{201} = -128 \equiv 22 \pmod{25}; \\ 2017^2 &\equiv 14 \pmod{25}. \end{aligned}$$

It follows $P \equiv 11 \pmod{25}$, by which two last digits of P are in the set $\{11, 36, 61, 86\}$. In other side, $P \equiv 1 \pmod{4}$. This implis $P \equiv 61 \pmod{100}$. Thus, the number 61 subjects to the question.

Question 8. Determine all real solutions x, y, z of the following system of equations

$$\begin{cases} x^3 - 3x &= 4 - y \\ 2y^3 - 6y &= 6 - z \\ 3z^3 - 9z &= 8 - x. \end{cases}$$

Solution. From $x^3 + y = 3x + 4$ it follows $x^3 - 2 - 3x = 2 - y$. Then

$$(x-2)(x+1)^2 = 2 - y \tag{1}$$

By $2y^3 - 4 - 6y = 2 - z$, we have

$$2(y-2)(y+1)^2 = 2 - z. \tag{2}$$

Similarly, by $3z^3 - 3 - 3 - 9z = 2 - x$ we have

$$3(z-2)(z+1)^2 = (2 - x). \tag{3}$$

Combining (1)-(2)-(3) we obtain

$$(x-2)(y-2)(z-2)\left((x+1)^2(y+1)^2(z+1)^2 + \frac{1}{6}\right) = 0.$$

Hence, $(x-2)(y-2)(z-2) = 0$. Comparing this with (1), (2) and (3), we find the unique solution $x = y = z = 2$.

Question 9. Prove that the equilateral triangle of area 1 can be covered by five arbitrary equilateral triangles having the total area 2.

Solution.

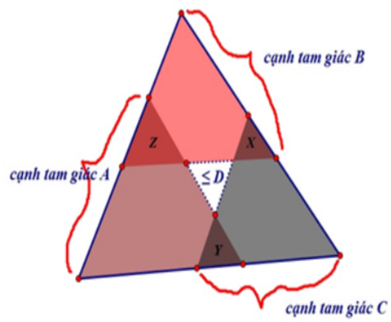

Figure 1: For Question 9

Let S denote the triangle of area 1. It is clearly that if $a \geq b$ then triangle of area a can cover triangle of area b. It suffices to consider the case when the areas of five small triangles are all smaller than 1. Let $1 \geq A \geq B \geq C \geq D \geq E$ stand for the areas. We will prove that the sum of side-lengths of B and C is not smaller than the side-length of triangle of area 1. Indeed, suppose $\sqrt{B} + \sqrt{C} < \sqrt{1} = 1$. It follows

$$2 = A + B + C + D + E < 1 + B + C + 2\sqrt{BC} = 1 + (\sqrt{B} + \sqrt{C})^2 < 2,$$

which is impossible.

We cover S by A, B, C as Figure 1. We see that A, B, C will have common parts, mutually. Suppose

$$X = B \cap C; \quad Y = A \cap C; \quad Z = A \cap B.$$

It follows

$$X + Y \leq C; \quad Y + Z \leq A; \quad Z + X \leq B.$$

We deduce A, B, C cover a part of area:

$$\begin{aligned} A + B + C - X - Y - Z &\geq A + B + C - \frac{1}{2}\big[(X+Y) + (Y+Z) + (Z+X)\big] \\ &\geq \frac{1}{2}(A + B + C) = 1 - \frac{D+E}{2} \geq 1 - D. \end{aligned}$$

Thus, D can cover the remained part of S.

Question 10. Find all non-negative integers a, b, c such that the roots of equations:

$$x^2 - 2ax + b = 0; \tag{1}$$
$$x^2 - 2bx + c = 0; \tag{2}$$
$$x^2 - 2cx + a = 0 \tag{3}$$

are non-negative integers.

Solution. We see that $a^2 - b, b^2 - c, c^2 - a$ are perfect squares. Namely,

$$a^2 - b = p^2; \quad b^2 - c = q^2; \quad c^2 - a = r^2.$$

There are two cases:
Case 1. $b = 0$. We derive that $b = c = 0$. Thus $(a, b, c) = (0, 0, 0)$ is unique solution.
Case 2. $a, b, c \neq 0$. We have $a^2 - b \leq (a-1)^2 = a^2 - 2a + 1$. This implies $b \geq 2a - 1$. Similarly, we can prove that $c \geq 2b - 1$, and $a \geq 2c - 1$. Combining three above inequalities we deduce $a + b + c \leq 3$. By simple computation we obtain $(a, b, c) = (1, 1, 1)$.

Question 11. Let S denote a square of the side-length 7, and let eight squares of the side-length 3 be given. Show that S can be covered by those eight small squares.

Solution.

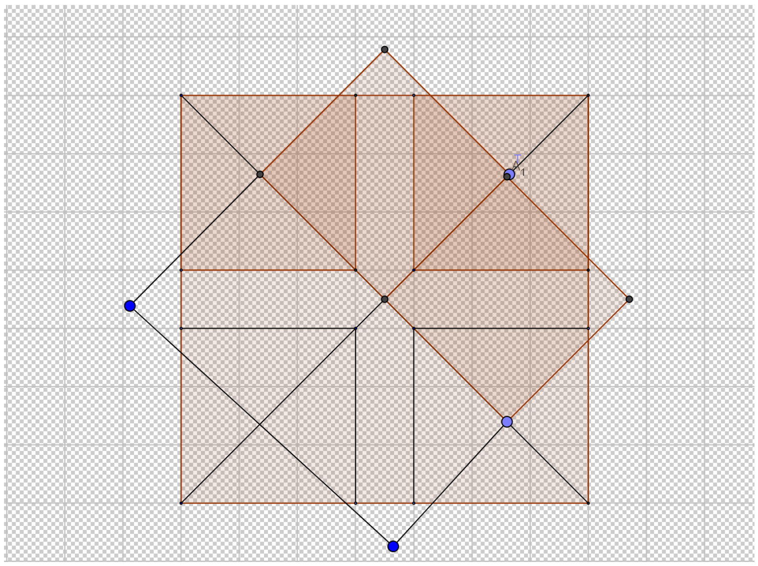

Figure 2: For Question 11

Figure 2 is a solution.

Question 12. Does there exist a sequence of 2017 consecutive integers which contains exactly 17 primes?

Solution. It is easy to see that there are more than 17 primes in the sequence of numbers $1, 2, 3, 4, \dots, 2017$. Precisely, there are 306 primes in that sequence. Remark that if the sequence

$$k+1, k+2, \dots, k+2017$$

was changed by the sequence

$$k, k+1, \dots, k+2016,$$

then the numbers of primes in the latter and former sequences are either equal, more or less by 1. In what follows, we say the such change *a shift back with 1 step.* First moment, we consider the sequence of 2017 consecutive integers:

$$2018!+2, 2018!+3, \dots 2018!+2018$$

which contain no prime. After 2018!+1 times shifts back, we obtain the sequence

$$1, 2, 3, 4, \dots, 2017.$$

The last sequence has 306 primes, while the first sequence has no prime. Reminding the above remark we conclude that there is a moment in which the sequence contains exactly 17 primes.

Question 13. Let a, b, c be the side-lengths of triangle ABC with $a+b+c=12$. Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

Solution. Put

$$x := \frac{b+c-a}{2}, y := \frac{c+a-b}{2}, z := \frac{a+b-c}{2}.$$

Then $x, y, z > 0$, and

$$x+y+z = \frac{a+b+c}{2} = 6,\ a = y+z, b = z+x, c = x+y.$$

We have

$$M = \frac{y+z}{2x} + \frac{4(z+x)}{2y} + \frac{9(x+y)}{2z} = \frac{1}{2}\left[\left(\frac{y}{x} + \frac{4x}{y}\right) + \left(\frac{z}{x} + \frac{9x}{z}\right) + \left(\frac{4z}{y} + \frac{9y}{z}\right)\right]$$
$$\geq \frac{1}{2}\left(2\sqrt{\frac{y}{x}.\frac{4x}{y}} + 2\sqrt{\frac{z}{x}.\frac{9x}{z}} + 2\sqrt{\frac{4z}{x}.\frac{9y}{z}}\right) = 11.$$

The equality occurs in the above if and only if

$$\begin{cases} \dfrac{y}{x} = \dfrac{4x}{y} \\ \dfrac{z}{x} = \dfrac{9x}{z} \\ \dfrac{4z}{y} = \dfrac{9y}{z}, \end{cases}$$

or

$$\begin{cases} y = 2x \\ z = 3x \\ 2z = 3y. \end{cases}$$

Since $x+y+z=6$ we receive $x=1,\ y=2,\ z=3$. Thus $\min S = 11$ if and only if $(a,b,c) = \big(5,4,3\big)$.

Question 14. Given trapezoid $ABCD$ with bases $AB \parallel CD$ $(AB < CD)$. Let O be the intersection of AC and BD. Two straight lines from D and C are perpendicular to AC and BD intersect at E, i.e. $CE \perp BD$ and $DE \perp AC$. By analogy, $AF \perp BD$ and $BF \perp AC$. Are three points E, O, F located on the same line?

Solution. Since E is the orthocenter of triangle ODC, and F is the orthocenter of triangle OAB we see that OE is perpendicular to CD, and OF is perpendicular to AB. As AB is parallel to CD, we conclude that E, O, F are straightly lined.

Question 15. Show that an arbitrary quadrilateral can be divided into nine isosceles triangles.

Solution. Figures 3, 4, and 5 shows some solution.

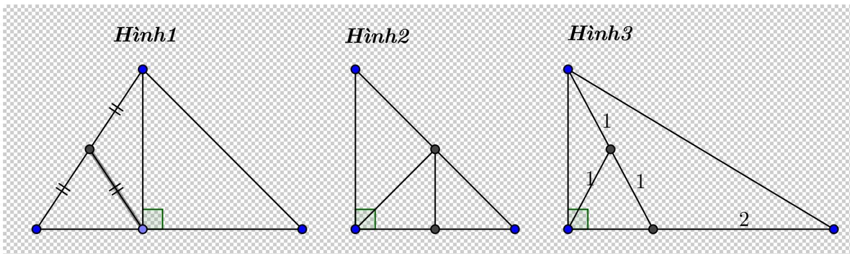

Figure 3: For Question 15

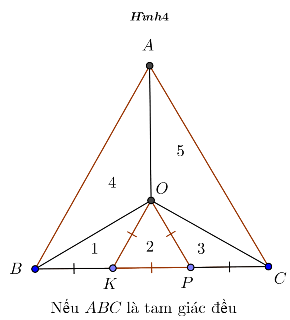

Figure 4: For Question 15

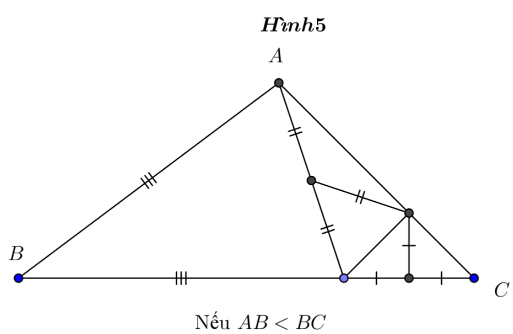

Figure 5: For Question 15

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. Suppose x_1, x_2, x_3 are the roots of polynomial $P(x) = x^3 - 4x^2 - 3x + 2$. The sum $|x_1| + |x_2| + |x_3|$ is
(A): 4 (B): 6 (C): 8 (D): 10 (E): None of the above.

Solution. The solution is (B).

Question 2. How many pairs of positive integers (x, y) are there, those satisfy the identity

$$2^x - y^2 = 4?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Solution. The solution is (A).

Question 3. The number of real triples (x, y, z) that satisfy the equation

$$x^4 + 4y^4 + z^4 + 4 = 8xyz$$

is
(A): 0; (B): 1; (C): 2; (D): 8; (E): None of the above.

Solution. The solution is (E).

Question 4. Let a, b, c be three distinct positive numbers. Consider the quadratic polynomial

$$P(x) = \frac{c(x-a)(x-b)}{(c-a)(c-b)} + \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + 1.$$

The value of $P(2017)$ is
(A): 2015 (B): 2016 (C): 2017 (D): 2018 (E): None of the above.

Solution. The solution is (D).

Question 5. Write 2017 following numbers on the blackboard:

$$-\frac{1008}{1008}, -\frac{1007}{1008}, \ldots, -\frac{1}{1008}, 0, \frac{1}{1008}, \frac{2}{1008}, \ldots, \frac{1007}{1008}, \frac{1008}{1008}.$$

One processes some steps as: erase two arbitrary numbers x, y on the blackboard and then write on it the number $x + 7xy + y$. After 2016 steps, there is only one number. The last one on the blackboard is
(A): $-\frac{1}{1008}$ (B): 0 (C): $\frac{1}{1008}$ (D): $-\frac{144}{1008}$ (E): None of the above.

Solution. The solution is (D).

Question 6. Find all pairs of integers a, b such that the following system of equations has a unique integral solution (x, y, z)

$$\begin{cases} x + y = a - 1 \\ x(y+1) - z^2 = b. \end{cases}$$

Solution. Write the given system in the form

$$\begin{cases} x + y + 1 = a \\ x(y+1) - z^2 = b. \end{cases} \tag{$*$}$$

System (*) is symmetric by $x, y+1$ and is reflect in z at 0 then the necessary condition for (*) to have a unique solution is $(x, y+1, z) = (t, t, 0)$. Putting this in (*), we find $a^2 = 4b$. Conversely, if $a^2 = 4b$ then

$$(x - (y+1))^2 + 4z^2 = (x + y + 1)^2 + 4z^2 - 4x(y+1) = a^2 - 4b = 0.$$

This implies the system has a unique solution

$$(x, y+1, z) = \left(\frac{a}{2}, \frac{a}{2}, 0\right).$$

Question 7. Let two positive integers x, y satisfy the condition $x^2 + y^2 \vdots 44$. Determine the smallest value of $T = x^3 + y^3$.

Solution. Note that $44 = 4 \times 11$ It follows $x^2 + y^2 \vdots 11$. We shall prove that $x \vdots 11$ and $y \vdots 11$. Indeed, if x and y are not divisible by 11 then by the Fermat's little theorem, we have

$$x^{10} + y^{10} \equiv 2 \pmod{11}. \tag{1}$$

On the other hand, since $x^2 + y^2 \vdots 11$ then $x^2 + y^2 \equiv 0 \pmod{11}$. It follows

$$x^{10} + y^{10} \equiv 0 \pmod{11},$$

which is not possible by (1). Hence, $x \vdots 11$ or $y \vdots 11$ and that follow $x \vdots 11$ and $y \vdots 11$ simultaneously (if $x \vdots 11$ then from $x^2 + y^2 \vdots 11$. It follows $y^2 \vdots 11$ and then $y \vdots 11$). In other side, we have $x^2 + y^2 \vdots 4$ and $x^2 \equiv 0, 1 \pmod{4}, y^2 \equiv 0, 1 \pmod{4}$. We then have $x^2 \equiv 0 \pmod{4}, y^2 \equiv 0 \pmod{4}$. It follows $x \vdots 2, y \vdots 2$. Since $(2, 11) = 1$, $x \vdots (22)$ and $y \vdots (22)$. Thus, $\min A = (22)^3 + (22)^3 = 21296$.

Question 8. Let a, b, c be the side-lengths of triangle ABC with $a + b + c = 12$. Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

Solution. Let $x = \dfrac{b+c-a}{2}, y = \dfrac{c+a-b}{2}, z = \dfrac{a+b-c}{2}$ then $x, y, z > 0$ and

$x + y + z = \dfrac{a+b+c}{2} = 6, a = y + z, b = z + x, c = x + y$. We have

$$M = \frac{y+z}{2x} + \frac{4(z+x)}{2y} + \frac{9(x+y)}{2z} = \frac{1}{2}\left[\left(\frac{y}{x} + \frac{4x}{y}\right) + \left(\frac{z}{x} + \frac{9x}{z}\right) + \left(\frac{4z}{y} + \frac{9y}{z}\right)\right]$$
$$\geq \frac{1}{2}\left(2\sqrt{\frac{y}{x}.\frac{4x}{y}} + 2\sqrt{\frac{z}{x}.\frac{9x}{z}} + 2\sqrt{\frac{4z}{y}.\frac{9y}{z}}\right) = 11.$$

The equality yields if and only if

$$\begin{cases} \dfrac{y}{x} = \dfrac{4x}{y} \\ \dfrac{z}{x} = \dfrac{9x}{z} \\ \dfrac{4z}{y} = \dfrac{9y}{z}. \end{cases}$$

Equivelently,

$$\begin{cases} y = 2x \\ z = 3x \\ 2z = 3y. \end{cases}$$

By simple computation we receive $x = 1, y = 2, z = 3$. Therefore, $\min S = 11$ when $(a, b, c) = \big(5, 4, 3\big)$.

Question 9. Cut off a square carton by a straight line into two pieces, then cut one of two pieces into two small pieces by a straight line, ect. By cutting 2017 times we obtain 2018 pieces. We write number 2 in every triangle, number 1 in every quadrilateral, and 0 in the polygons. Is the sum of all inserted numbers always greater than 2017?

Solution. After 2017 cuts, we obtain 2018 n-convex polygons with $n \geq 3$. After each cut the total of all sides of those n-convex polygons increases at most 4. We deduce that the total number of sides of 2018 pieces is not greater than 4×2018. If the side of a piece is k_j, then the number inserted on it is greater or equal to $5 - k_j$. Therefore, the total of all inserted numbers on the pieces is greater or equal to

$$\sum_j (5 - k_j) = 5 \times 2018 - \sum_j k_j \geq 5 \times 2018 - 4 \times 2018 = 2018 > 2017.$$

The answer is positive.

Question 10. Consider all words constituted by eight letters from $\{C, H, M, O\}$. We arrange the words in an alphabet sequence. Precisely, the first word is CCCCCCCC, the second one is CCCCCCCH, the third is CCCCCCCM, the fourth one is CCCCCCCO,. . . , and the last word is OOOOOOOO.

a) Determine the 2017^{th} word of the sequence?

b) What is the position of the word HOMCHOMC in the sequence?

Solution. We can associate the letters C, H, M, O with four numbers $0, 1, 2, 3$, respectively. Thus, the arrangement of those words as a dictionary is equivalent to arrangement of those numbers increasing.

a) Number 2017 in quaternary is $\{133201\}_4 = \{00133201\}_4 \sim CCHOOMCH$.

b) The word $HOMCHOMC$ is corresponding to the number $\{13201320\}_4$ which means the number 13201320 in quaternary. Namely,

$$\{13201320\}_4 = 4^7 + 3 \times 4^6 + 2 \times 4^5 + 0 \times 4^4 + 1 \times 4^3 + 3 \times 4^2 + 2 \times 4 + 0.$$

A simple computation gives $\{13201320\}_4 = 30840$. Thus, the word $HOMCHOMC$ is 30840^{th} in the sequence.

Question 11. Let ABC be an equilateral triangle, and let P stand for an arbitrary point inside the triangle. Is it true that

$$\left|\widehat{PAB} - \widehat{PAC}\right| \geq \left|\widehat{PBC} - \widehat{PCB}\right|?$$

Solution. If P lies on the symmetric straightline Ax of ΔABC, then

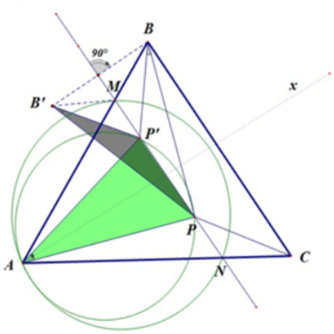

Figure 6: For Question 11

$$\left|\widehat{PAB} - \widehat{PAC}\right| = \left|\widehat{PBC} - \widehat{PCB}\right|.$$

We should consider other cases. Let P' denote the symmetric point of P with respect to Ax. The straightline PP'' intersects AB and AC at M and N, respectively. Choose B' that is symmetric point of B with respect to MN. Then

$$\left|\widehat{PAB} - \widehat{PAC}\right| = \widehat{PAP'},$$

and

$$\left|\widehat{PBC} - \widehat{PCB}\right| = \widehat{PBP'} = \widehat{PB'P'}.$$

We will prove that

$$\widehat{PAP'} \geq \widehat{PB'P'}. \qquad (*)$$

Indeed, consider the circumscribed circle (O) of the equilateral triangle AMN. Since

$$\widehat{MB'N} = \widehat{MBN} \leq \widehat{MBC} = \widehat{MAN} = 60^0,$$

B' is outside (O). Consider the circumscribed circle (O') of the equilateral triangle APP'. It is easy to see that (O') inside (O), by which B' is outside (O'). Hence, $\widehat{PAP'} \geq \widehat{PB'P'}$. The inequality (*) is proved.

Question 12. Let (O) denote a circle with a chord AB, and let W be the midpoint of the minor arc AB. Let C stand for an arbitrary point on the major arc AB. The tangent to the circle (O) at C meets the tangents at A and B at points X and Y, respectively. The lines WX and WY meet AB at points N and M, respectively.

Does the length of segment NM depend on position of C?

Solution. Let T be the common point of AB and CW. Consider circle (Q) touching XY at C and touching AB at T.

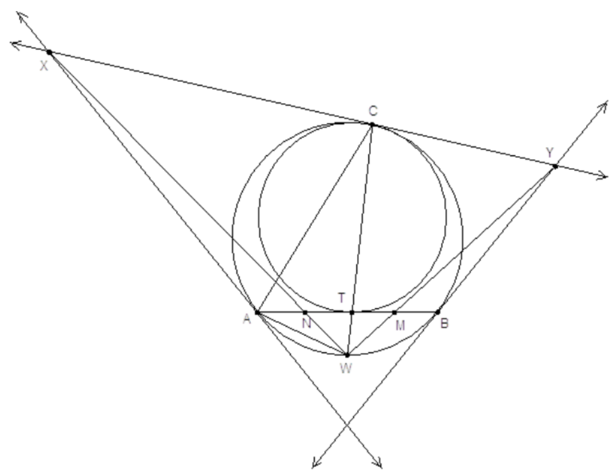

Figure 7: For Question 12

Since

$$\widehat{ACW} = \widehat{WAT} \quad \left(= \frac{1}{2}\overset{\frown}{AW} = \frac{1}{2}\overset{\frown}{WB}\right)$$

and $\widehat{AWT} = \widehat{CWA}$, we obtain that $\Delta AWT, \Delta CWA$ are similar triangles. Then

$$WA^2 = WT \times WC.$$

It is easy to see that WX is the radical axis of A and (Q), thus it passes through the midpoint N of segment AT. Similarly, WY passes through the midpoint M of segment BT. We deduce $MN = \dfrac{AB}{2}$.

Question 13. Let ABC be a triangle. For some $d > 0$ let P stand for a point inside the triangle such that

$$|AB| - |PB| \geq d, \text{ and } |AC| - |PC| \geq d.$$

Is the following inequality true

$$|AM| - |PM| \geq d,$$

for any position of $M \in BC$?

Solution. Note that AM always intersects PB or PC of ΔPBC. Without loss of generality, assume that AM has common point with PB. Then $ABMP$ is a convex quadrilateral with diagonals AM and PB.

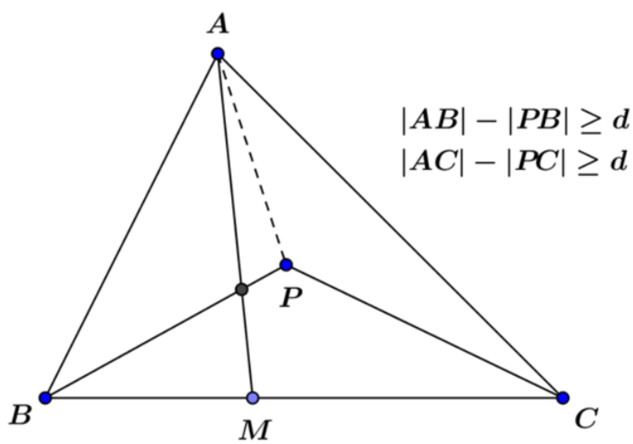

Figure 8: For Question 13

It is known that for every convex quadrilateral, we have

$$|AM| + |PB| \geq |AB| + |PM|,$$

that follows

$$|AM| - |PM| \geq |AB| - |PB| \geq d.$$

Question 14. Put

$$P = m^{2003}n^{2017} - m^{2017}n^{2003}, \quad \text{where} \quad m, n \in \mathbb{N}.$$

a) Is P divisible by 24?
b) Do there exist $m, n \in \mathbb{N}$ such that P is not divisible by 7?

Solution. We have

$$P = m^{2003}n^{2013}\big(n^{14} - m^{14}\big) = m^{2003}n^{2013}\big(n^7 - m^7\big)\big(n^7 + m^7\big).$$

It is easy to prove P is divisible by 8, and by 3.

b) It suffices to chose m, n such that the remainders of those divided by 7 are not 0 and distinct. For instance, $m = 2$ and $n = 1$.

Question 15. Let S denote a square of side-length 7, and let eight squares with side-length 3 be given. Show that it is impossible to cover S by those eight small squares with the condition: an arbitrary side of those (eight) squares is either coincided, parallel, or perpendicular to others of S.

Solution. Let $ABCD$ be the square S, and M, N, P, Q be the midpoints of sides of S, and O is the center of S. Consider nine points: $A, B, C, D, M, N, P, Q, O$. Each square of the side-length 3 satisfied the condition cover at most one of those nine points. The proof is complete.