Cubes and related problems

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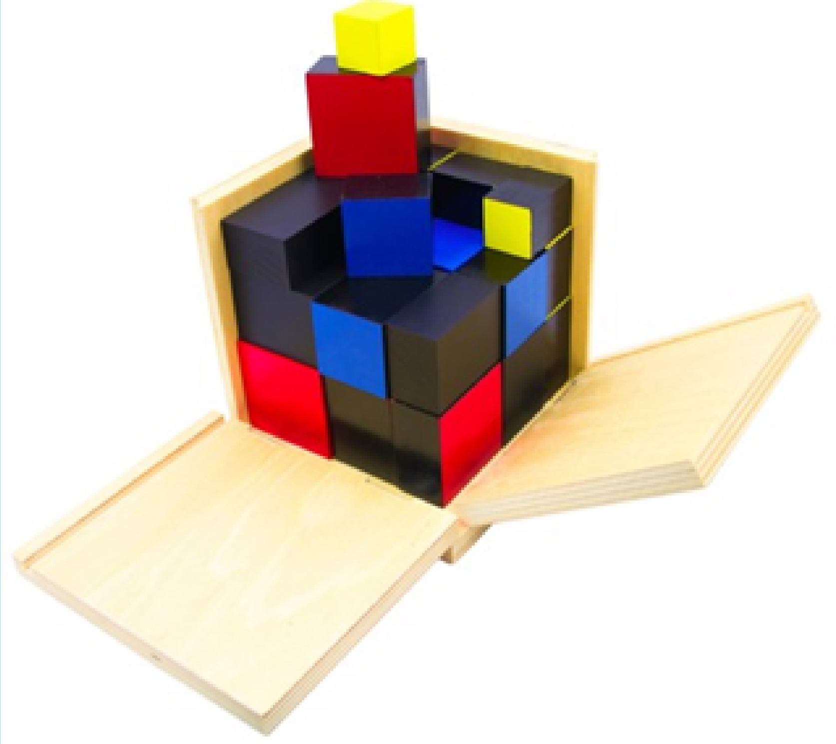

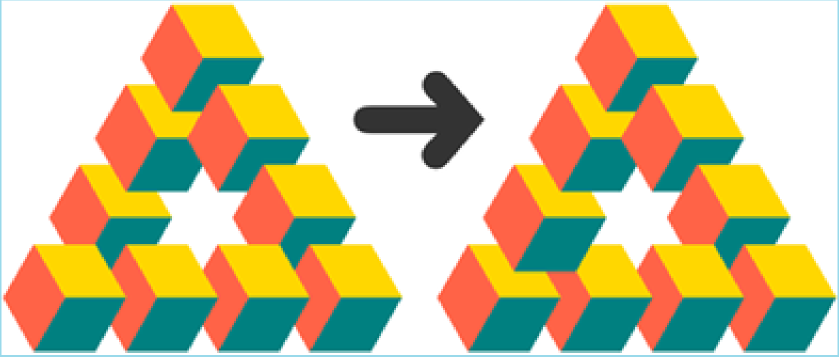

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Trong kỷ yếu này chúng tôi cũng vinh dự được giới thiệu một bài báo về "Khối lập phương và các vấn đề liên quan". Đây là một đề tài thú vị, hấp dẫn và luôn luôn trẻ. Bài báo này tổng kết một số các kết quả quan trọng chúng tôi đang trong quá trình thực hiện.

Những bài toán liên quan đến khối lập phương thì nhiều, nhưng những tài liệu tổng kết và phân loại về nó thì lại có không nhiều, một phần chắc do hạn chế về công cụ thể hiện cũng như khả năng xuất bản mầu, vì đề tài này cần nhiều hình vẽ và mầu sắc. Trong đề án đang thực hiện này, song song với mục đích tổng hợp, phân loại và hệ thống hóa các kết quả quan trọng của các nghiên cứu tập trung về khối lập phương, chúng tôi còn đưa mục tiêu: Dễ dạy – Dễ học – Và khơi nguồn cảm hứng cho các phát triển tiếp theo.

Trong giảng dậy chúng ta cũng cảm thấy vấn đề phẳng và không gian có mốc phân cách chính là khối lập phương. Làm bạn được với cách nhìn và suy nghĩ lập phương thì việc chiếm lĩnh nhãn quan và phương pháp của toán hiện đại trở thành gần gũi.

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NHỮNG GÌ KHÔNG DẠY ĐƯỢC LÚC LỚN THÌ PHẢI DẠY NGAY TỪ NHỎ!
Cũng là động lực thôi thúc chúng tôi bắt tay vào nghiên cứu và xây dựng giáo trình thí điểm này.

Chúng tôi hy vọng được sự quan tâm và đóng góp tích cực của các đồng nghiệp, các bạn yêu toán để sớm có một tài liệu học bổ ích.
Đây là bài báo bằng tiếng Anh. Chúng tôi sẽ hoàn chỉnh tuyển tập bằng tiếng Việt.

Xin cảm ơn các bạn!

Thư từ xin chuyển về địa chỉ: Loiscenter@gmail.com

Mục lục

KẾ HOẠCH NGHIÊN CỨU

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3. Cutting a cube by a plane
4. Coloring cubes
5. Paths in a cube

Part B

6. Geometric transformations with cubes
7. Filling cubes or spaces with cube-like shapes
8. Probability, dice games
9. Calculations related to cubes
10. Other problems

Cubes and related problems

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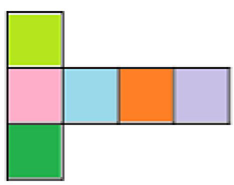
Matematical problems around cubes are very old but remain interesting to this day. They pose exciting challenges to learners and educators alike. These problems set apart good students from the average. In this document, I try to summarize and categorize characteristic problems revolving around cubes. Reference material can be found at [1], [2], [3], [4], [5].
A number of problems were collected from Hungarian mathematics camps.

1 Spread the cube on the plane

1.1. We will show that 11 flat nets can be drawn for a cube. Cases that can be rotated to overlap are not counted as different.

We can list the separate nets.

a) There are four squares in a line. Because if there were 5 squares in one line, it could not be folded into a cube as two faces would overlap. This case provides six different nets from 1 to 6.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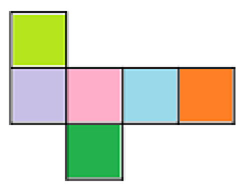
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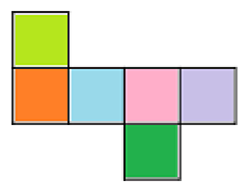
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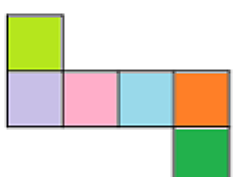
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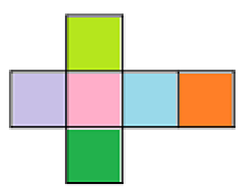
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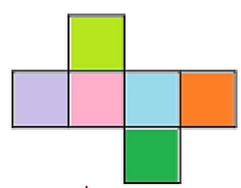
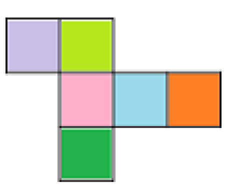

6)

b) There are exactly 3 squares in a row. We get the 3−line nets $(2, 3, 1)$ of the squares in Fig. 7, 8, 9 and a 2−line $(3, 3)$ net in Figure 10.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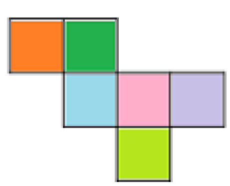
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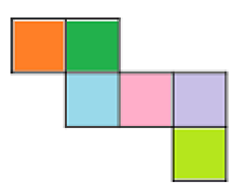
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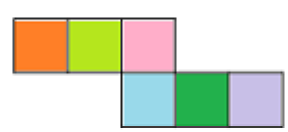
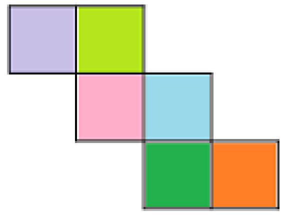

10)

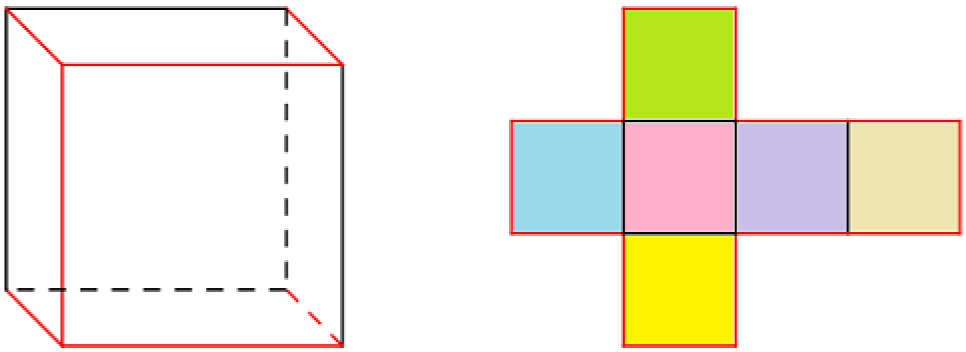
c) Finally, there is a 3−line $(2, 2, 2)$ square as shown in Figure 11.



11)

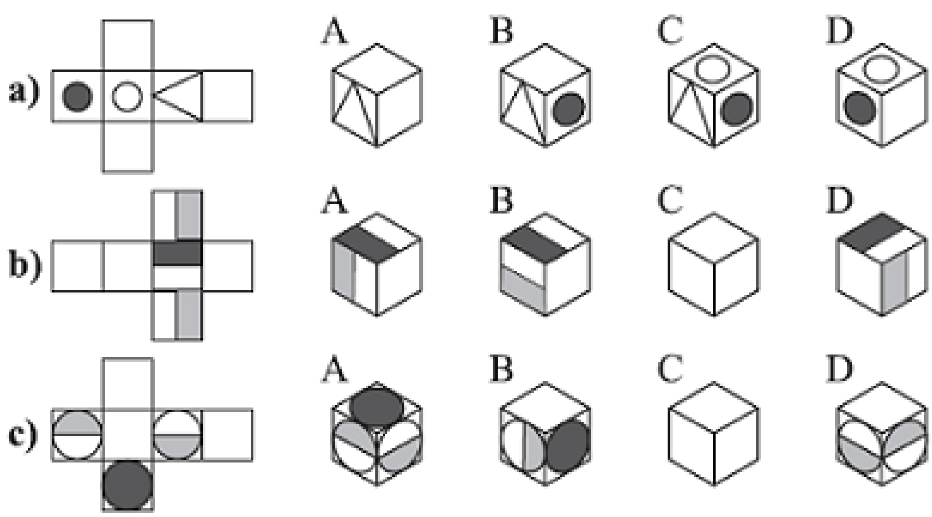
he above net we can fold to get the cube.

1.2 Cut a cube along the edges to get the cube net.
Guide: On the picture.

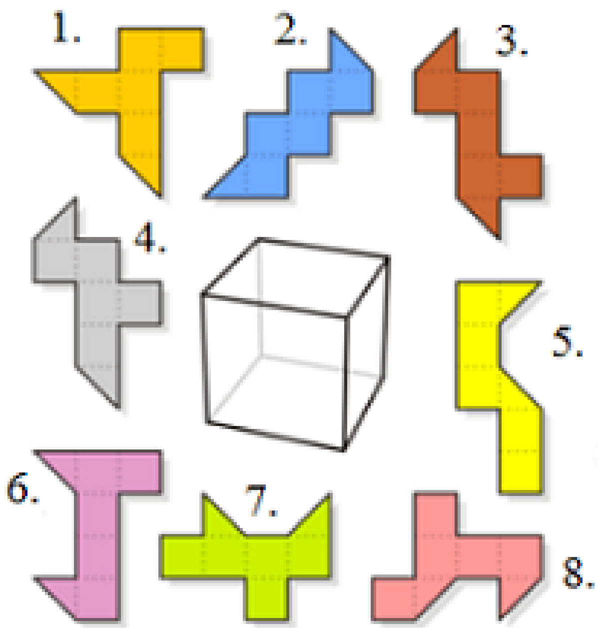


Exercise 1.1. Cut the cube along the corresponding edges to get the cubes from 1 to 11.

Exercise 1.2. Of A, B, C, D which cube matches the net showed on the left?



Exercise 1.3. On the picture, each net is made of 5 squares and 2 triangles. Which nets can be folded into a cube?



Exercise 1.4. You have a square paper. Have should you cut one piece from it so that it can be folded into a cube of the largest volume?

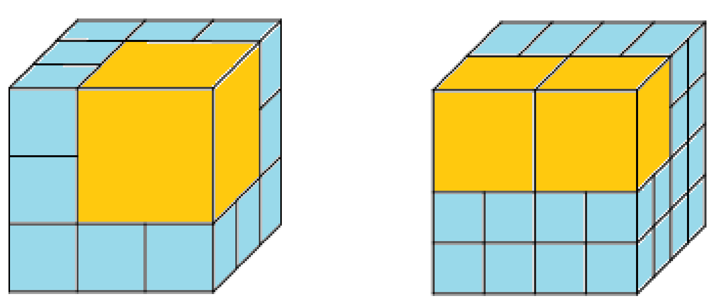
You can find more exercises in Reference [3], [4].

2 Cubes

Problem 2.1. Can you cut a cube into 20 cubes? Can you cut it into 50 cubes?

Solution: Possible for both cases.

With the dimensions $2 \times 2 \times 2$, $3 \times 3 \times 3$, and $4 \times 4 \times 4$, we can create 8, 27, and 64 cubes. In reverse, it is $64 \to 1,\ 27 \to 1,\ 8 \to 1$.



Therefore:

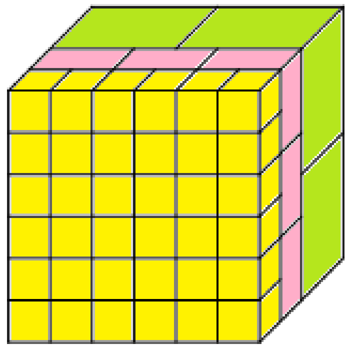
$$20 = 27 - (8 - 1) = 27 - 7.$$
$$50 = 64 - 2 \cdot (8 - 1) = 64 - 2 \cdot 7.$$

Problem 2.2. Can you cut a cube into 48 smaller cubes?

Answer: Possible $27 + 3 \cdot (8 - 1) = 27 + 3 \cdot 7 = 48$.

Problem 2.3. Can you cut the cube into 49 cubes?

Answer: Possible.



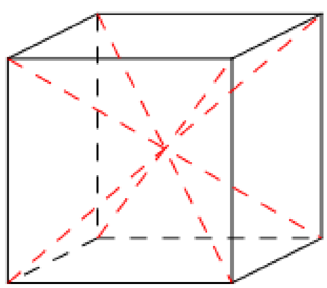
Use a cube with faces of 6 units ($6 \times 6 \times 6$). Divide it into $6 \times 6 = 36$ unit ($1 \times 1 \times 1$) cubes, $\cdot 3 \times 3 = 9$ cubes, ($2 \times 2 \times 2$) and $2 \times 2 = 4$ cubes ($3 \times 3 \times 3$). Then $36 + 9 + 4 = 49$.

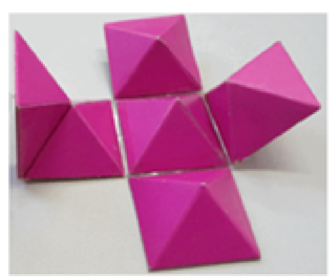
Exercise 2.1. For what n value can you cut a cube into n small cubes?

Problem 2.4. Is it possible to cut a cube into identical pyramids? Can it be cut into 3 identical pyramids?

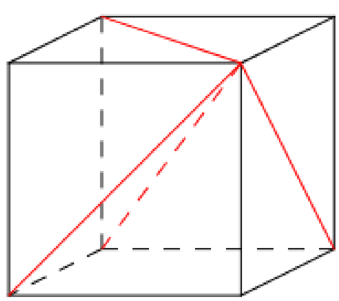
Solution:

a) Possible.
The center of the cube connected to all the vertices makes six pyramids.





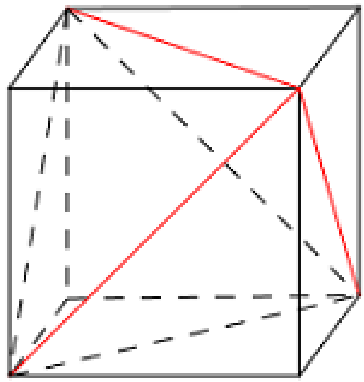
b) Possible.



Connect the vertices on one face with the other vertices on other faces and continue until you get 3 identical pyramids.

Problem 2.5. A cube is cut into tetrahedrons. At least how many tetrahedrons do we get?

Solution: Each face of the cube is a square, so it should be divided into at least two parts. Select two opposing faces (of the cube). There are four triangles without two being in the same tetrahedron. The tetrahedral faces are one of these four triangles and they will have a total volume of no more than $\frac{2}{3}$, which indicates that we need more than four tetrahedrons.

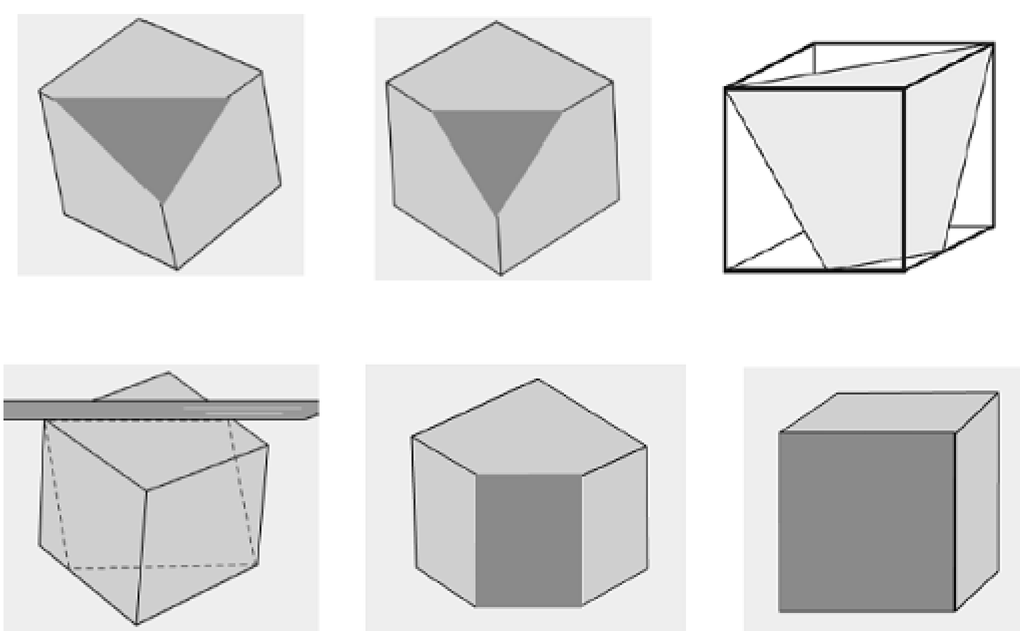


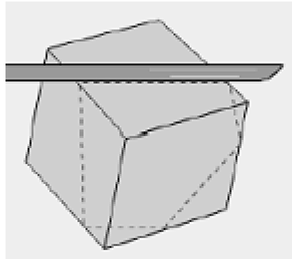
In case it can be cut into 5 tetrahedrons: select 4 vertices without two being on the same face of the cube. We get a tetrahedron. With the four faces of the tetrahedron being joined with four from the other face of the surface, we get 5 tetrahedrons that completely cover the cube (Figure).

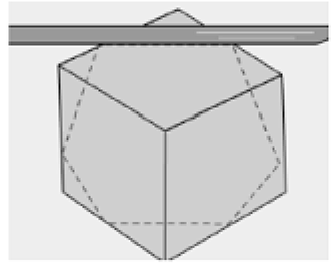
You can find more exercises in Reference [2], [3].

3 Cut the cube by a plane

3.1 Possible cross section of the cube with a plane.





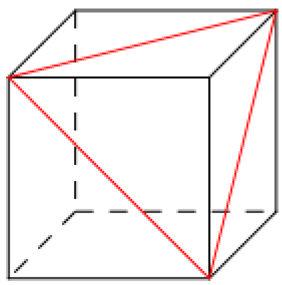


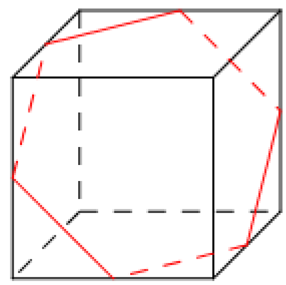
Problem 3.1. Can the cross section of a cube and a plane be. . .

a) . . . an equilateral triangle?

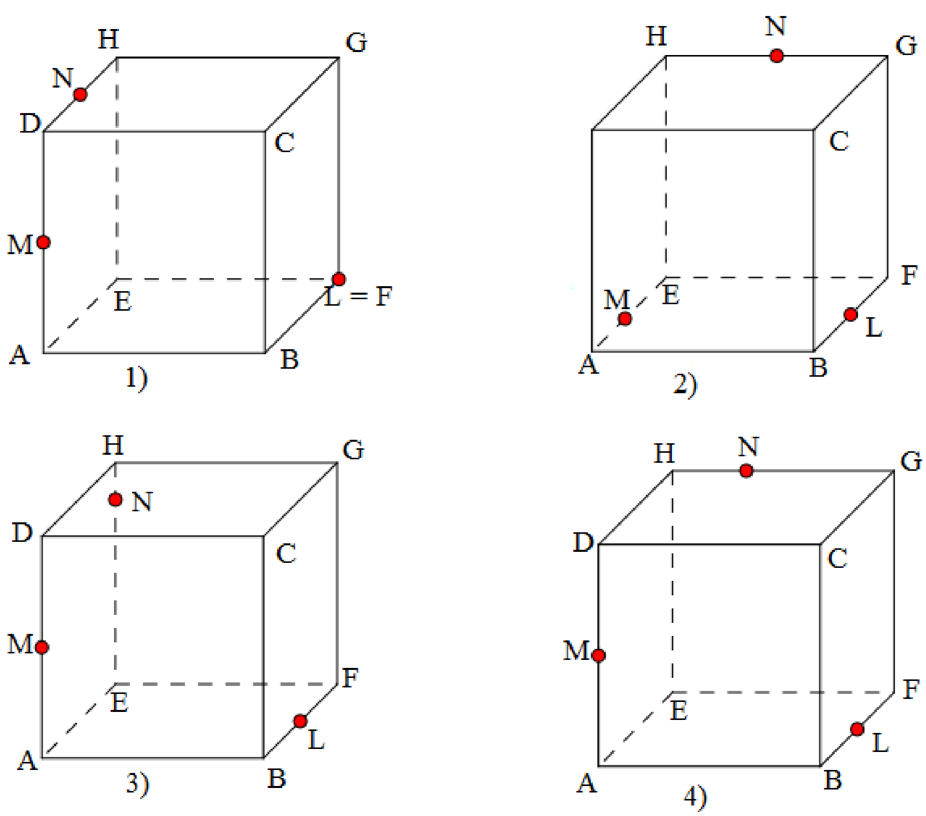
b) . . . a hexagon?

Solution: Illustration.





Exercise 3.1. For the cube $ABCDEFGH$, the plane (p) goes through 3 points M, N and L (figure). Construct the cross section created by the $p-$plane and the cubes.

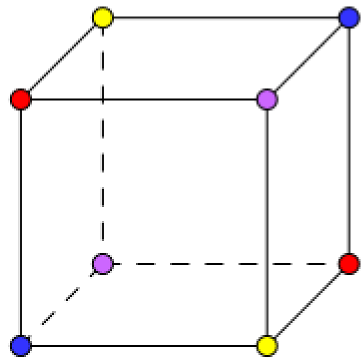


You can find more exercises in Reference [4].

4 Paint Color cubes

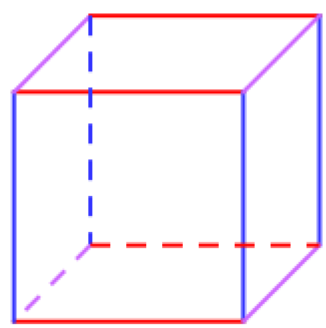
Problem 4.1. At least how many different colors can we paint the vertices of the cube so that connected vertices (those lying along the same edge) have different colors?

Solution: We need at least 4 colors.



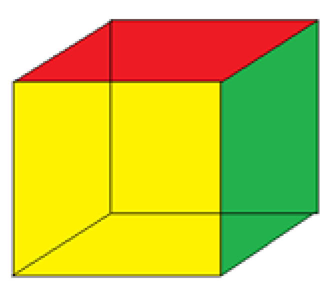
Problem 4.2. At least how many colors can you paint the edges of *a* cube so that edges sharing the same vertex have different colors?

Solution: We need 3 colors.



Problem 4.3. At least how many colors do you need to paint the faces of a cube so that the faces on the same edge have different colors?

Solution: We need at least 3 colors.



Problem 4.4. In how many ways can we paint the faces of a cube in black and white so that each face only has one color? Cubes that can be rotated to overlap do not count as different.

Solution: With two different colors, there are 8 ways to paint the cube. If we allow unicolor cubes, there are two more ways, which makes it a total of 10 solutions for a maximum of two colors. Let us have a look at the different ways and put them into a table.

Obviously, there is only 1 cube which has $0, 1, 5$ or 6 red faces.

If two faces are red, those two faces can be either adjacent or opposite to each other. These are two options and naturally there are two similar options with 4 red and 2 blue faces.

In case we have three red faces, let us add one red face to the previous scenario. If the two red faces are opposite each other, there is only one way of painting the third face red. If the two red faces are adjacent, then there is one more way of painting the cube, with each red face joining the two other red faces on two sides. This are also two options.

Number of black sides	0	1	2	3	4	5	6
Number of white sides	6	5	4	3	2	1	0
Number of way	1	1	2	2	2	1	1

A total of 10 ways.

Problem 4.5. In how many ways can we paint the faces of a cube in 6 different colors so that each face has only one color? Cubes that can be rotated to overlap do not count as different.

Solution: There are 30 ways.
First we paint one face red and place the cube standing on this face (this face will be covered). This leaves 5 faces. There are 5 ways to paint the top face. The other four faces can be rotate, so there are $4 \cdot 3 \cdot 2 \cdot \frac{1}{4} = 6$ ways. This makes a total of $5 \cdot 6 = 30$ ways.

Problem 4.6. In how many ways can we paint the faces of a cube in 5 colors so that each face has only one color? Cubes that can be rotated to overlap do not count as different.

Solution: There are 75 ways.
There can be exactly 2 faces of the same color. If they are opposite faces, the other 4 faces can be painted in $\frac{4!}{4}$ ways. However, the faces of same color can also be swapped by rotation, which leaves $\frac{6}{2} = 3$ different color combinations.

If the two faces of the same color share a common edge, then the remaining 4 faces can be painted in $4! = 24$ different ways, but these contain doubles. This leaves $\frac{24}{2} = 12$ ways, which brings it to a total of $12 + 3 = 15$ different ways.

Let us not forget to multiply this number with 5 (5 different colors) to get $5 \times 15 = 75$ different ways.

Exercise 4.1. In how many ways can we paint the faces of a cube in 3 colors so that each face has only one color? Cubes that can be rotated to overlap do not count as different.

Answer: There are 30 ways.

Exercise 4.2. In how many ways can we paint the faces of a cube in 4 colors so that each face has only one color? Cubes that can be rotated to overlap do not count as different.

Answer: There are 68 ways.

Exercise 4.3. In how many ways can we paint the faces of a cube in n different colors so that each face has only one color? Cubes that can be rotated to overlap do not count as different.

Answer: $\frac{1}{24} \times (n^6 + 3 \cdot n^4 + 12 \cdot n^3 + 8 \cdot n^2)$.

Problem 4.7. In how many ways can we paint the faces of a cube in three colors (red, blue, yellow) so that there are two faces of blue, two of red, and two of yellow? Cubes that can be rotated to overlap do not count as different.

Solution: There are 6 ways.

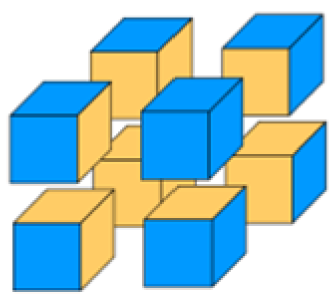
a) If the two red faces are opposite, then either the other pairs are opposite too (one way) or they are adjacent (one way). This makes it $1 + 1 = 2$ ways.

b) When the two red faces are adjacent (they have a common edge), there are two possibilities. We either have one pair of opposite sides with the same color, or all opposing sides are of different color. There are 2 ways of painting the faces in each scenario, which makes it $2 + 2 = 4$ ways.

Thus, in total, there are $2 + 4 = 6$ different ways to paint the cube.

Problem 4.8. Can we paint the faces of 8 small cubes in two colors so that we can build 2 different $2 \times 2 \times 2$ cubes from them?

Solution: A $2 \times 2 \times 2$ cube has $6 \times 4 = 24$ external faces and the same number of internal faces. For this reason only the painting seen on the figure can work. This is suitable as we can build the yellow cube by turning around the small cubes.



Problem 4.9.

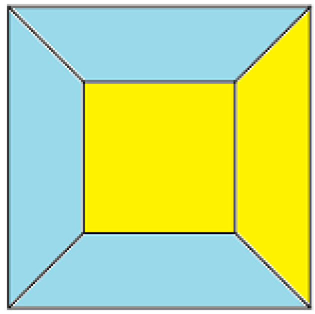
a) If we paint the faces of 27 small cubes in two colors in any way so that each face is one color, can we build a $3 \times 3 \times 3$ cube of uniform color?

b) Can we do it with the condition that each cube has an equal number of faces of each color?

Solution:

a) Impossible, paint 13 cubes red, 14 blue. Both groups will be present on the surface of the $3 \times 3 \times 3$ cube.

b) Impossible, in this case, the tube can be painted in the following way (the top face that is not visible is yellow).



None of the vertices of the cube are surrounded by the same color. Therefore, it is not always possible to build a cube of uniform color.

Exercise 4.4. There are 27 small cubes. What is the largest number of faces that we can paint red so that we cannot build a $3 \times 3 \times 3$ cube of uniform color.

Exercise 4.5.

a) There is a $3 \times 3 \times 3$ cube. Vertex A is red (while all the others are white). Tam wants to move the red cube to the opposite vertex of the cube. She can only change the layer (horizontal or vertical) containing the red cube with a neighbouring one. Cam wants to stop her from reaching her goal and uses magic to decide how many changes Tam is allowed to make (anywhere between $5-10$). Can Tam do it?

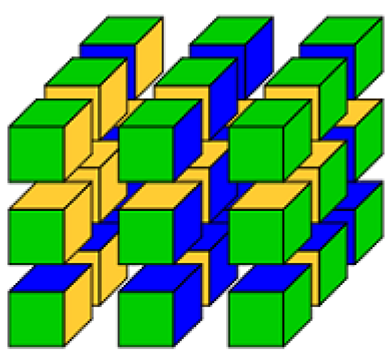
b) If Tam's goal is to move the red cube to a neighbouring position, can she do it?

Problem 4.10. Can we paint 27 small cubes in 3 different colors so that they can be joined into 3 different $3 \times 3 \times 3$ cubes of uniform color?

Solution: Let the three colors be blue, yellow and green. For a large cube we need $6 \cdot 9 = 54$ blue faces. We need the same number of tiles for the other two colors too.

Altogether we need to paint 162 faces. This is equal to the number of faces on 27 smaller cubes $(6 \cdot 27)$, thus there might be a way to solve this exercise if we paint each and every face.

Let us have a look at the blue faces. We need 8 cubes that have 3 blue faces to serve as the vertices. We also need 12 cubes with two blue faces to place between the vertices on the edges, and 6 cubes that have one blue face to go in the middle of each larger face.



Let us denote these with $B_3,\ B_2,\ B_1$. We need the same from the other two colors, let us denote them similarly.

Thus, we need 8 $B_3,\ Y_3,\ G_3$; 12 $B_2,\ Y_2,\ G_2$ and 6 $B_1,\ Y_1,\ G_1$. This can be made in the following way:
1 cube of $B_3,\ Y_3$; 1 cube of $Y_3,\ G_3$; 1 cube of $G_3,\ B_3$;
6 cubes of $B_2,\ Y_2,\ G_2$; 6 cubes of $B_3,\ Y_2,\ G_1$; 6 cubes of $Y_3,\ G_2,\ B_1$; 6 cubes of $G_3,\ B_2,\ Y_1$.

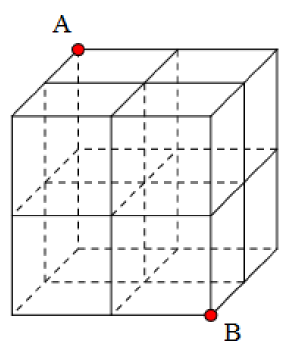
We also need to consider whether these cubes can actually painted this way and whether they can be rotated in the correct configuration, but we leave this up to the Reader.

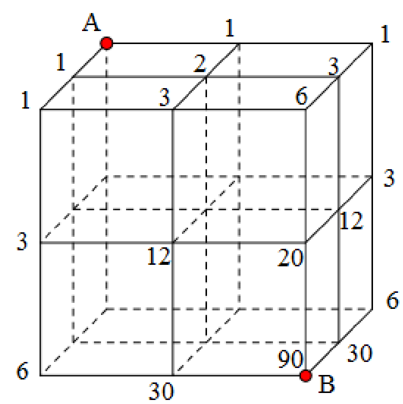
You can find more exercises in Reference [3], [5].

5 Paths in the cube

Problem 5.1. On a $3\times3\times3$ cube, how many paths go from point A to point B in a way that one can only move from node to node and can only go either right, left or down (there is also a node at the center of the cube)?

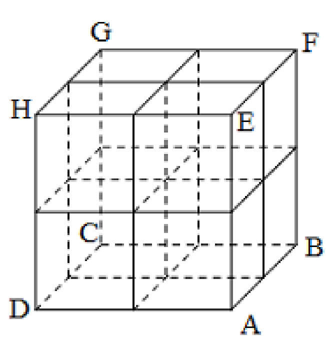
Solution: There are a total of 90 ways.
The counting method is illustrated in the figure.

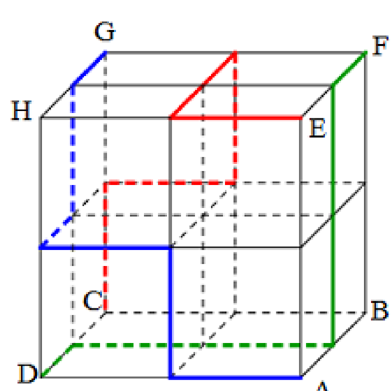




Problem 5.2. Consider a cube formed by a $3\times3\times3$ grid. The vertices are $ABCD$ and $EFGH$ where (A,E); (B,F); (C,G) and (D,H) are pairs of opposing vertices. Show that it is possible to join the edges of the cube so that the three pairs of contiguous vertices have distinct paths leading to each other without having any common vertices, but there is no such connection for 4 pairs of vertices.

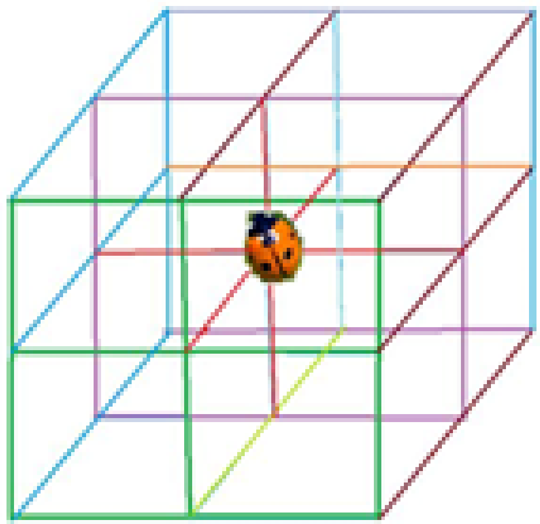
Guide: Refer to the drawing.





(A,G); (D,F); (C,E)

Exercise 5.1. Orange is always thinking about math. She is going for a stroll around his glass villa. The paths are straight lines that connect the nodes. Before going, Orange wants to draw a path so the she only has to go through each node once.
Can Orange draw such a path? Please help her find the solution!



You can find more exercises in Reference [5].

6 Other exercises

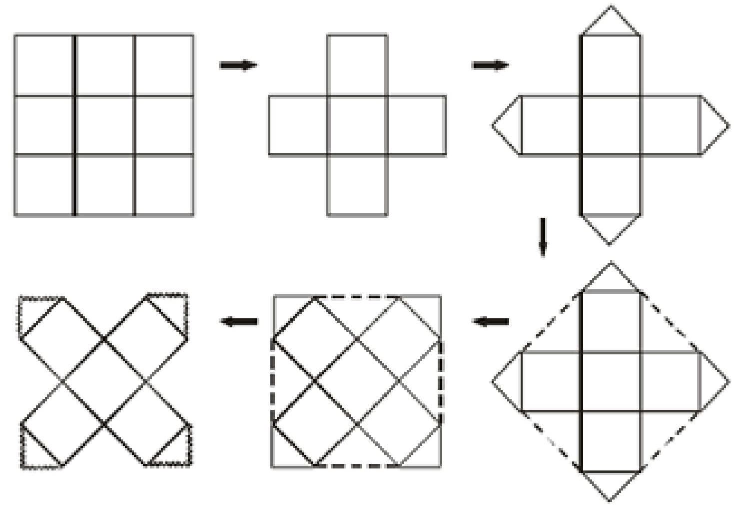
Exercise 6.1. An and Binh are playing the following dice game. On An's dice, there are the following numbers: $4, 6, 10, 18, 20, 22$. Binh's dice has $3, 9, 13, 15, 17, 25$. Both players will roll their own dice and the winner is whoever gets the larger number. Who has the advantage in this game?

Exercise 6.2. An and Binh play a dice game. Both An and Binh's dice have positive integers on their faces. Both players roll their dice and the winner is whoever gets the larger number. Is it true that if the average of the six numbers on An's dice is greater than that on Bin's, then An will have a better chance of winning than Binh?

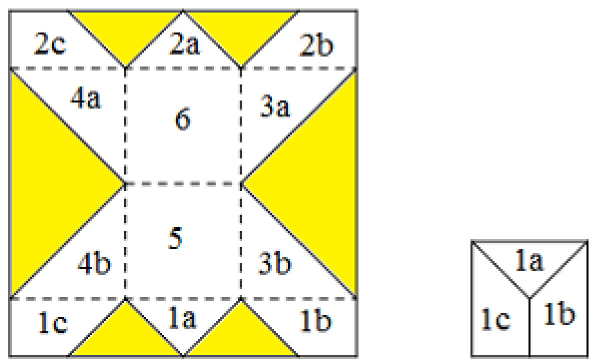
Exercise 6.3. An and Binh play a dice game. There are three empty dice on the table. An writes the numbers from 1 to 18 on the dice and picks one. Binh chooses one of the remaining dice. Both players roll their dice and the winner is whoever gets the larger number (the third dice does not play a role.) Who has the advantage, An or Binh?

Problem 6.1. From a 3×3 square cut out the frame of a cube in one piece so that the cuts are either parallel or perpendicular to each other.

Solution:
Look at the illustration for the cutting sequence. Looking at the top row, it might seem like we are cutting along the lines of the original grid, which is false. Eventually, we cut the frame of the cube out of a $2\sqrt{2} \times 2\sqrt{2}$ square.



We cut at a 45^0 angle compared to the original grid, but the lines of folding are parallel to the grid lines.



(Collected at Hungarian camps in 2005)

Exercise 6.4. All 8 edges of a pyramid are equal. Is it possible to use 6 of such identical pyramids to build a cube in a way that their vertices that are parallel to their base touch?

Exercise 6.5. We have a cubic shaped cake with an even chocolate frosting on the top and the sides. How do we carve it so that everybody gets the same amount of cake and chocolate frosting?
How can we do this if we have $n = 2, 3, 4, 5$ people?
Can the exercise be solved for any natural n number?

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Hà nội, 15/03/2018.
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